Probabilistic Combinatorics (Fall 2023)

Thursday, September 7, 2023 (in-person exam)

Solve any four of the problems below.

Problem 1. Let $k \geq 3$ be fixed and let $p = c/n$. Show that there exists $\theta = \theta(c,k) > 0$ such that w.h.p. all vertex sets S with $|S| \leq \theta n$ contain fewer than $k|S|/2$ edges. Deduce that w.h.p. either the k-core of $G_{n,p}$ is empty or it has size at least θn.

(Recall: Given a positive integer k, the *k-core* of a graph $G = (V, E)$ is the largest set $S \subseteq V$ such that the minimum degree in the vertex-induced subgraph $G[S]$ is at least k.)

Problem 2. Let $v_1, v_2, \ldots v_n$ be n vectors from $\{\pm 1\}^n$ chosen uniformly and independently. Let M_n be the largest pairwise dot product in absolute value: i.e

$$M_n = \max_{i \neq j} |v_i \cdot v_j| \,.$$

Prove that

$$\frac{M_n}{2\sqrt{n \ln n}} \to 1 \,,$$

in probability as $n \to \infty$.

Recall that to show $\frac{M_n}{2\sqrt{n \ln n}}$ converges to 1 in probability, we must show that for every $\epsilon > 0$,

$$\Pr\Big[\big|\frac{M_n}{2\sqrt{n \ln n}} - 1\big| > \epsilon\Big] \to 0 \text{ as } n \to \infty \,.$$

Problem 3. Prove that for every k, r ($k \geq 2$) there exists a family $\mathcal{F}$ of k-sets so that $|A \cap B| \leq 1$ for all $A, B \in \mathcal{F}$ and for any r-coloring of the underlying points some $A \in \mathcal{F}$ is monochromatic.

Hint: Use the *deletion/alteration method*

Problem 4. Let k be sufficiently large. Let $\mathcal{F}$ be a k-uniform family of sets and suppose that no element belongs to more than k sets of $\mathcal{F}$. Then show that it is possible to color the elements using $r = \lfloor k/\log k \rfloor$ colors so that every member of $\mathcal{F}$ has at most $\lceil 2e \log k \rceil$ elements of the same color.

Problem 5. An urn initially contains one white and one black ball. At each stage a ball is drawn and is then replaced in the urn along with another ball of the same color. Let Z_n denote the fraction of balls in the urn that are white after the nth replication.

(a) Show that $\{Z_n, n \geq 1\}$ is a martingale.

(b) Show that the probability that the fraction of white balls in the urn is ever as large as 3/4 is at most 2/3.