Proabilistic Combinatorics Basic Exam

January 26, 2023

3 hour. Closed book. Of course, all answers should be justified.

1. Show that if n is sufficiently large then there is a graph G on n vertices such that the minimum degree of n is at least $n/2$ and there is no dominating set in G with fewer than $\frac{\log(n)}{10}$ vertices.

2. Let X be the number of isolated edges (i.e. components that consist of a single edge) in $G_{n,p}$. Set
$$p = \frac{\log(n) + \log\log(n) + f(n)}{2n},$$
and assume $|f(n)| < \log\log(n)$. Prove that
$$\mathbb{P}(X > 0) \to \begin{cases} 0 & \text{if } f(n) \to \infty \\ 1 & \text{if } f(n) \to -\infty. \end{cases}$$

3. Let Y be the number of isolated vertices in $G_{n,p}$. Now set
$$p = \frac{\log(n) + C}{n},$$
where C is an absolute constant. Determine
$$\lim_{n\to\infty} \mathbb{P}(Y = 0),$$
and justify your answer.

4. Let G be a graph and let p denote the probability that a random subgraph of G obtained by picking each edge of G with probability $1/2$, independently, is connected and spanning. Let q denote the probability that a random 2-coloring of G, where the color of each edge is chosen randomly and independently to be either red or blue, has the property that both the red graph and the blue graph are connected and spanning. Is $q \leq p^2$?

5. Consider a coloring of the integers using r colors. We say that a subset of the integers is *rainbow* if it contains elements of all r colors. Fix finite sets of integers A and B such that $|A| = a$ and $|B| = b$. Prove that if

$$4rb^2e^{-b/r} < 1$$

then there is a coloring of integers with the property that for every $x \in A$ the set $x + B = \{x + y : y \in B\}$ is rainbow.

6. (a) State the definition of a coupling of Markov chains.

   (b) Let (X_t, Y_t) be a coupling of Markov chains such that for some $\alpha < 1$ and some $t_0 > 0$ the coupling time

$$\tau_{\text{couple}} = \min\{t \geq 0 : X_t = Y_t\}$$

satisfies $Pr(\tau_{\text{couple}} \leq t_0) \geq \alpha$ for all pairs of initial states x, y. Prove

$$E[\tau_{\text{couple}}] \leq \frac{t_0}{\alpha}.$$