Probabilistic Combinatorics: Sample Basic Exam

1. Which functions \( p = p(n) \) have the property
   \[
   Pr(G_{n,p} \text{ has no isolated vertices}) \to 1?
   \]

2. Let \( \mathcal{F} \) be a collection of \( k \)-element subsets of a set \( X \) with the property that no element of \( X \) is in more than \( k \) sets. (In other words, \( \mathcal{F} \) is a \( k \)-uniform hypergraph with maximum degree \( k \).) Prove that if \( k \) is sufficiently large then there is a coloring of \( X \) in \( \lceil k/\log k \rceil \) colors such that no set in \( \mathcal{F} \) has more than \( 10 \log k \) elements with the same color.

3. Let \( \mathcal{H} \) be a collection of subsets of \( [n] = \{1, 2, \ldots, n\} \) with the following property: If \( A \) and \( B \) are distinct sets in \( \mathcal{H} \) then the symmetric difference of \( A \) and \( B \) contains at least \( n/3 \) elements. How many sets can the collection \( \mathcal{H} \) contain?
   Prove upper and lower bounds on the maximum \( |\mathcal{H}| \). (You may assume \( n \) is large.)

4. We choose a set of integers \( A \subseteq [n] \) uniformly at random. Let the random variable \( X \) be the number of arithmetic progressions of length \( (\log_2 n)/10 \) in \( A \). Prove that there is a sequence of integers \( f(n) \) such that \( f(n) \to \infty \) and \( X = (1 + o(1))f(n) \) with high probability. (Of course, you should try to make the \( o(1) \) term as small as possible.)

5. We have a collection of \( n \) bins. A sequence of \( n \) balls arrive one at a time. When ball \( i \) arrives a bin \( A_i \) is chosen uniformly at random. If \( A_i \) is an empty bin then the \( i^{th} \) ball is placed in \( A_i \). If bin \( A_i \) is not empty then a second bin \( B_i \) is chosen at random and the \( i^{th} \) ball is placed in that bin (regardless of the number balls already in the bin).
   Let \( Y \) be the number of empty bins after all \( n \) balls have been placed in bins. Find a real number \( \alpha \) such that \( Y = (\alpha + o(1))n \) with high probability. (You do not need to prove concentration; simply find the number \( \alpha \).)