Probabilistic Combinatorics: Sample Basic Exam

1. Which functions $p = p(n)$ have the property

$$Pr(G_{n,p} \text{ has no isolated vertices}) \to 1?$$

2. Let $\mathcal{F}$ be a collection of k-element subsets of a set X with the property that no element of X is in more than k sets. (In other words, $\mathcal{F}$ is a k-uniform hypergraph with maximum degree k.) Prove that if k is sufficiently large then there is a coloring of X in $\lfloor k/\log k \rfloor$ colors such that no set in $\mathcal{F}$ has more than $10 \log k$ elements with the same color.

3. Let $\mathcal{H}$ be a collection of subsets of $[n] = \{1, 2, \ldots, n\}$ with the following property: If A and B are distinct sets in $\mathcal{H}$ then the symmetric difference of A and B contains at least $n/3$ elements. How many sets can the collection $\mathcal{H}$ contain?

   Prove upper and lower bounds on the maximum $|\mathcal{H}|$. (You may assume n is large.)

4. We choose a set of integers $A \subseteq [n]$ uniformly at random. Let the random variable X be the number of arithmetic progressions of length $(\log_2 n)/10$ in A. Prove that there is a sequence of integers $f(n)$ such that $f(n) \to \infty$ and $X = (1 + o(1))f(n)$ with high probability. (Of course, you should try to make the $o(1)$ term as small as possible.)

5. We have a collection of n bins. A sequence of n balls arrive one at a time. When ball i arrives a bin A_i is chosen uniformly at random. If A_i is an empty bin then the i^{th} ball is placed in A_i. If bin A_i is not empty then a second bin B_i is chosen at random and the i^{th} ball is placed in that bin (regardless of the number balls already in the bin).

   Let Y be the number of empty bins after all n balls have been placed in binds. Find a real number α such that $Y = (\alpha + o(1))n$ with high probability. (You do not need to prove concentration; simply find the number α.)