Basic Examination: MEASURE AND INTEGRATION
January 21, 2025, 5:30pm-8:30pm

• This test is closed book: no notes, Internet sources, or other aids are permitted.

• You have 3 hours. The exam has a total of 5 questions and 100 points.

• You may use without proof standard results from the syllabus which are independent of the question asked. You must, however, clearly state the results you are using.

1. [15 points] Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $f_n, g_n, f, g : X \to \mathbb{R}$, $n \in \mathbb{N}$, be measurable functions such that $f_n \to f$ in measure and $g_n \to g$ in measure. Prove that if $\mu(X) < \infty$ then $f_n g_n \to fg$ in measure.

2.

(i) [5 points] State Hölder's Inequality.

(ii) [15 points] Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) < \infty$, and let $f \in L^\infty(X)$. Prove that

$$\lim_{p\to\infty} ||f||_{L^p} = ||f||_{L^\infty}.$$

3.

(i) [5 points] State Lebesgue Dominated Convergence Theorem.

(ii) [15 points] Evaluate

$$\lim_{n\to\infty} \int_0^\infty \frac{n^{1/4} e^{-x^2 n}}{1+x^2}\, dx.$$

4. Let $(X, \mathcal{M}, \mu)$ be a measure space, and suppose that $X = \cup_{n=1}^\infty E_n$, where $\{E_n\}$ is a collection of pairwise disjoint measurable sets such that $\mu(E_n) < \infty$ for all $n \in \mathbb{N}$. Define ν on $\mathcal{M}$ by

$$\nu(E) := \sum_{n=1}^\infty \frac{\mu(E \cap E_n)}{2^n(\mu(E_n) + 1)}.$$

(i) [10 points] Prove that ν is a finite measure on $(X, \mathcal{M})$.

(ii) [10 points] Prove that $\nu << \mu$ and $\mu << \nu$.

(iii) [10 points] Find $\frac{d\mu}{d\nu}$ and $\frac{d\nu}{d\mu}$.

5.[15 points] Let $(X, \mathcal{M})$ be a measurable space, let $\lambda : \mathcal{M} \to \mathbb{R}$ be a signed measure on $(X, \mathcal{M})$ and let μ, ν be positive measures on $(X, \mathcal{M})$ such that $\mu = \nu - \lambda$. Prove that $\nu \geq \lambda^+$ and $\mu \geq \lambda^-$.