Basic Examination: MEASURE AND INTEGRATION
August 26, 2024, 5:30pm-8:30pm

• This test is closed book: no notes, Internet sources, or other aids are permitted.

• You have 3 hours. The exam has a total of 5 questions and 100 points.

• You may use without proof standard results from the syllabus which are independent of the question asked. You must, however, clearly state the results you are using.

1. [20 points]

Let $(X, \mathcal{M}, \mu)$ be a measure space, let μ be a probability measure, and let $f \in L^p_\mu(X;\mathbb{R})$, for $p \in (1,\infty)$. Consider the function $\theta : [1,p] \to \mathbb{R}$ given by

$$\theta(r) := \ln ||f||^r_{L^r_\mu(X;\mathbb{R})}.$$

Is θ convex? Prove or disprove.

2.

(i) [8 points] Let $(X, \mathcal{M}, \mu)$ be a positive measure space with μ σ-finite, and let $f : X \to [0,\infty)$ be an integrable function. Prove that

$$\int_X f\,d\mu = (\mu \otimes \mathcal{L}^1)(\{(x,y) \in X \times [0,\infty) : 0 \le y \le f(x)\}),$$

where $\mathcal{L}^1$ is the 1-dimensional Lebesgue measure on $\mathbb{R}$, and $\mu \otimes \mathcal{L}^1$ stands for the product measure of μ and $\mathcal{L}^1$.

(ii) [22 points] Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $f : X \to \mathbb{R}$ be a measurable function. Let $p \in [1,\infty)$. Prove that $|f|^p$ is μ-integrable if and only if the function

$$t \in [0,\infty) \mapsto t^{p-1}\,\mu(\{x \in X : |f(x)| > t\})$$

is integrable with respect to $\mathcal{L}^1$, and one has

$$\int_X |f|^p\,d\mu = p\int_0^\infty t^{p-1}\,\mu(\{x \in X : |f(x)| > t\})\,dt.$$

3. Consider the Lebesgue measure space $(\mathbb{R}, \mathcal{M}, \mathcal{L}^1)$, where $\mathcal{M}$ is the σ-algebra of Lebesgue measurable sets. Let ν be the counting measure on $\mathcal{M}$.

(i) [12 points] Is $\mathcal{L}^1 << \nu$? Prove or disprove.

Is there a measurable function $f : \mathbb{R} \to [0,\infty]$ such that for all $A \in \mathcal{M}$

$$\mathcal{L}^1(A) = \int_A f\,d\nu?$$

Prove or disprove.

(ii) [13 points] Show that ν does not have a Lebesgue Decomposition with respect to $\mathcal{L}^1$. Why does the Lebesgue Decomposition Theorem fail?

4. [10 points]

Let $(X, \mathcal{M}, \mu)$ be a measure space, let μ be a probability measure, and let $f_n : X \to \mathbb{R}$ be μ-integrable functions, $n \in \mathbb{N}$.

Is this statement true or false? If

$$\sup_{n \in \mathbb{N}} ||f_n||_{L^1} < \infty$$

then

$$\liminf_{n \to \infty} |f_n(x)| < \infty \, \text{a.e.} x \in X.$$

Prove or disprove

5. [15 points]

Let $(X, \mathcal{M}, \mu)$ be a measure space, let μ be a probability measure, and let $f_n : X \to \mathbb{R}$ be μ-integrable functions, $n \in \mathbb{N}$. Assume that there exists $M > 0$ such that for all $n \in \mathbb{N}$

$$\int_X \left| f_n(x) - \int_X f_n \, d\mu \right|^2 \, d\mu(x) \leq M \int_X |f_n| \, d\mu.$$

Prove that if

$$\limsup_{n \to \infty} \int_X |f_n| \, d\mu = \infty$$

then

$$\limsup_{n \to \infty} |f_n(x)| = \infty \, \text{a.e.} x \in X.$$