Measure Theory Basic Exam.

Exam date: 2023-08-29. Time limit: 3 hours.

- *This is a closed book test. You may not use any reference material or internet sources during the test.*
- *You may not seek or receive assistance.*
- *You may use standard results from the syllabus, unless explicitly stated otherwise.*
- *Good luck* ⌣̈.

1. Let μ be a positive measure on (X, Σ) and $p, q, r \in [1, \infty]$ with $p < q < r$. If $f \in L^p(X) \cap L^r(X)$, must $f \in L^q(X)$? If yes, prove it. If no find a counterexample.

2. Let μ, ν be two finite measures on (X, Σ). Suppose μ is a finite positive measure and ν is absolutely continuous with respect to μ. Must there exist finite constants C_1, C_2 (independent of μ and ν) such that
$$\|\nu\|_{\mathrm{TV}} \leqslant C_1 \Big\| \frac{d\nu}{d\mu} \Big\|_{L^1(\mu)} \qquad \text{and} \qquad \Big\| \frac{d\nu}{d\mu} \Big\|_{L^1(\mu)} \leqslant C_2 \|\nu\|_{\mathrm{TV}}\,?$$
Prove your answer. (Here $\|\nu\|_{\mathrm{TV}}$ denotes the total variation norm of ν.)

3. Let μ be a finite positive measure on (X, Σ) and $f \in L^1(X, \mu)$ be such that $\ln|f| \in L^1(X, \mu)$. Decide whether or not
$$\lim_{p \to 0} \frac{1}{p} \Big(\int_X (|f|^p - 1)\, d\mu \Big)$$
exists, and prove your answer.

4. (a) True or false: For every $d \in \mathbb{N}$, there exists a finite constant C such that for every $\alpha > 0$ and every positive measure μ we have
$$\big| \{x \in \mathbb{R}^d \mid M\mu(x) > \alpha\} \big| \leqslant \frac{C}{\alpha} \mu(\mathbb{R}^d)\,.$$
Here $M\mu$ denotes the maximal function of μ, and μ is a positive Borel measure on $\mathbb{R}^d$. If true, prove it. If false, find a counter example.

   (b) True or false: For every $d \in \mathbb{N}$, there exists a finite constant C such that for every $f \in L^1(\mathbb{R}^d)$ we have
$$\|Mf\|_{L^1} \leqslant C \|f\|_{L^1}\,.$$
As before, Mf denotes the maximal function of $f \in L^1(\mathbb{R}^d)$.

5. Let (X, Σ, μ) and (Y, τ, ν) be two measure spaces. Suppose μ and ν are both finite, positive measures. Let $\Sigma \otimes \tau$ be the σ-algebra generated by the set of all sets of the form $A \times B$ with $A \in \Sigma$ and $B \in \tau$. Show that there exists a measure π on $(X \times Y, \Sigma \otimes \tau)$ such that $\pi(A \times B) = \mu(A)\nu(B)$ for every $A \in \Sigma$, $B \in \tau$.