1. Let $\mu$ be a positive measure on $(X, \Sigma)$ and $p, q, r \in [1, \infty]$ with $p < q < r$. If $f \in L^p(X) \cap L^r(X)$, must $f \in L^q(X)$? If yes, prove it. If no find a counterexample.

2. Let $\mu, \nu$ be two finite measures on $(X, \Sigma)$. Suppose $\mu$ is a finite positive measure and $\nu$ is absolutely continuous with respect to $\mu$. Must there exist finite constants $C_1, C_2$ (independent of $\mu$ and $\nu$) such that

$$\|\nu\|_{TV} \leq C_1 \left\| \frac{d\nu}{d\mu} \right\|_{L^1(\mu)} \quad \text{and} \quad \left\| \frac{d\nu}{d\mu} \right\|_{L^1(\mu)} \leq C_2 \|\nu\|_{TV}.$$ 

Prove your answer. (Here $\|\nu\|_{TV}$ denotes the total variation norm of $\nu$.)

3. Let $\mu$ be a finite positive measure on $(X, \Sigma)$ and $f \in L^1(X, \mu)$ be such that $\ln|f| \in L^1(X, \mu)$. Decide whether or not

$$\lim_{p \to 0} \frac{1}{p} \left( \int_X (|f|^p - 1) \, d\mu \right)$$

exists, and prove your answer.

4. (a) True or false: For every $d \in \mathbb{N}$, there exists a finite constant $C$ such that for every $\alpha > 0$ and every positive measure $\mu$ we have

$$\left| \left\{ x \in \mathbb{R}^d \mid M_\mu(x) > \alpha \right\} \right| \leq \frac{C}{\alpha} \mu(\mathbb{R}^d).$$

Here $M_\mu$ denotes the maximal function of $\mu$, and $\mu$ is a positive Borel measure on $\mathbb{R}^d$. If true, prove it. If false, find a counter example.

(b) True or false: For every $d \in \mathbb{N}$, there exists a finite constant $C$ such that for every $f \in L^1(\mathbb{R}^d)$ we have

$$\|Mf\|_{L^1} \leq C \|f\|_{L^1}.$$ 

As before, $Mf$ denotes the maximal function of $f \in L^1(\mathbb{R}^d)$.

5. Let $(X, \Sigma, \mu)$ and $(Y, \tau, \nu)$ be two measure spaces. Suppose $\mu$ and $\nu$ are both finite, positive measures. Let $\Sigma \otimes \tau$ be the $\sigma$-algebra generated by the set of all sets of the form $A \times B$ with $A \in \Sigma$ and $B \in \tau$. Show that there exists a measure $\pi$ on $(X \times Y, \Sigma \otimes \tau)$ such that $\pi(A \times B) = \mu(A)\nu(B)$ for every $A \in \Sigma$, $B \in \tau$. 