Measure Theory Basic Exam.

Exam date: 2022-08-30. Time limit: 3 hours.

- *This is a closed book test. You may not use any reference material or internet sources during the test.*
- *You may not seek or receive assistance.*
- *You may use standard results from the syllabus, unless explicitly stated otherwise.*
- *Good luck* ☺.

1. Let (μ_n) be a sequence of positive measures on (X, Σ) be such that $\mu_{n+1}(E) \geqslant \mu_n(E)$ for every $E \in \Sigma$. Let $\mu(E) = \lim_{n\to\infty} \mu_n(E)$. Is μ necessarily a measure on (X, Σ)? If yes, prove it. If no, find a counter example.

2. Let X be a compact metric space, and μ be a finite positive Borel measure. Prove that μ is regular. That is, for any $A \in \mathcal{B}(X)$ show that $\mu(A) = \sup\{\mu(K) \mid K \subseteq A\,,\ \text{and } K \text{ is compact}\}$.

3. Suppose μ is a finite signed measure on (X, Σ). Show that there exists a set $P \in \Sigma$ with $\mu(P) \geqslant \mu(X)$ such that $\mu(E) \geqslant 0$ for every $E \subseteq P$.

   Note: Even though this is a standard result in your syllabus, please provide a complete proof here. Do not use the Radon-Nikodym or Hahn-Decomposition in your proof. You may, however, use other standard facts about measures that are independent of these results and this problem.

4. Let μ be a finite Borel measure on $\mathbb{R}^d$, and λ be the Lebesgue measure. If μ and λ are mutually singular then show
$$\lim_{r\to 0} \frac{\mu(B(x,r))}{\lambda(B(x,r))} = \infty\,, \qquad \text{for } \mu\text{-almost every } x \in \mathbb{R}^d\,.$$

   Note: Even though this is a standard result in your syllabus, please provide a complete proof here. You may however use without proof standard lemmas that were proved independently of this fact in standard references.

5. Let μ be a positive, σ-finite measure on (X, Σ), and let $p, q \in (1, \infty)$ be such that $\frac{1}{p} + \frac{1}{q} > 1$. Let $K\colon X \times X \to \mathbb{R}$ be measurable with respect to the product σ algebra $\sigma(\Sigma \times \Sigma)$. Suppose there are finite constants C, D such that
$$\sup_{y\in X} \int_X |K(x,y)|^p \, d\mu(x) = C^p\,, \qquad \text{and} \qquad \sup_{x\in X} \int_X |K(x,y)|^p \, d\mu(y) = D^p\,.$$

   Given $g \in L^q(X, \Sigma)$, define the function f by
$$f(x) \stackrel{\text{def}}{=} \int_X K(x,y) g(y) \, d\mu(y)\,.$$

   Explicitly find $r \in [1, \infty]$ such that $f \in L^r(X, \Sigma)$. Moreover, explicitly find an upper bound for $\|f\|_{L^r}$ in terms of p, q, C, D and $\|g\|_{L^q}$. Prove your answer.