Basic Examination: Measure and Integration January 2022

- This test is **closed book**: no notes or other aids are permitted.
- The exam has a total of 5 questions and 100 points (20 each).
- You have 3 hours. You must scan and upload your solutions to the Box folder within 15 minutes afterwards.
- You may use without proof *standard* results from the syllabus (not homework) which are independent of the question asked, unless explicitly instructed otherwise. You must, however, **clearly** state the result you are using.

Below, if not stated explicitly, $(X, \mathcal{F}, \mu)$ is a measure space, $L_p = L_p(X, \mathcal{F}, \mu)$ is the standard L_p space ($p \in [1, \infty]$) and m is Lebesgue measure.

1. Let $f \in L^1(\mathbb{R})$. Prove that $F(x) := \sum_{n=-\infty}^{\infty} f(n+x)$ converges absolutely, for m-a.e. $x \in [0,1]$.

2. Let $p \in [1, \infty)$. Suppose $f_n \in L^p(X)$ for each $n \in \mathbb{N}$, that $f_n(x) \to f(x)$ as $n \to \infty$ for μ-a.e. $x \in X$, and assume

$$\int |f_n|^p \, d\mu \to \int |f|^p \, d\mu.$$

Prove that $\|f_n - f\|_p \to 0$ as $n \to \infty$. (Hint: modify the proof of the dominated convergence theorem, after showing $h_n := 2^{-p}|f_n - f|^p \leq g_n := |f_n|^p + |f|^p$.)

3. Let $p \in (1, \infty)$ and let $u : \mathbb{R} \to [0, \infty)$ be continuous with $\|u\|_p = \left(\int |u(x)|^p \, dm(x) \right)^{1/p} < \infty$. Let $\alpha, \beta > 0$, and define

$$v_n(x) = \sum_{k=1}^{n} \alpha^k u(\beta^k x) \quad \text{for each } n \in \mathbb{N}, \text{ and} \quad v(x) = \sum_{k=1}^{\infty} \alpha^k u(\beta^k x).$$

Assume $\beta^{1/p} > \alpha$. Find a constant $C > 0$ independent of u such that necessarily $\|v\|_p \leq C\|u\|_p$.

4. Let $X = [0,1]$, and suppose λ is a finite Borel measure on $X \times X$ such that $\lambda \ll m \times m$, with

$$m(A) = \lambda(A \times X) = \lambda(X \times A) \quad \text{for all Borel } A \subset [0,1].$$

Is it true or false that necessarily $\lambda = m \times m$? Prove or provide a counterexample.

5. Let μ and ν be two Borel measures on the real interval $X = [0,1]$ such that

$$c_n = \int x^n \, d\mu(x) = \int x^n \, d\nu(x) < \infty \qquad \text{for every integer } n \geq 0.$$

(i) Prove that $\mu(I) = \nu(I)$ for each open interval $I \subset X$.
(ii) Infer that $\mu = \nu$.