# Measure Theory Basic Exam.

2021-02-09

- *This is a closed book test. You may not use any reference material or internet sources during the test.*
- *You may not seek or receive assistance.*
- *You may use standard results from the syllabus, unless explicitly stated otherwise.*
- *Good luck* $\ddot\smile$.

1. Let $q_1, q_2, \dots$ be an enumeration of the rational numbers, and define $f\colon \mathbb{R} \to \mathbb{R}$ by $f(x) = \displaystyle\sum_{k=1}^{\infty} \frac{e^{-|x-q_k|}}{2^k \sqrt{|x-q_k|}}$.
   (a) Is $f$ finite almost everywhere? Prove or disprove it.
   (b) Does there exist $a, b \in \mathbb{R}$ with $a < b$ such that $\displaystyle\int_a^b f(x)^2 \, dx < \infty$? Prove or disprove it.
2. Let $\gamma \in (0,1)$ and $R\colon [0,\infty) \to [0,\infty)$ be a continuous function such that $\lim_{t\to\infty} t^\gamma R(t) = c_0 \in (0,\infty)$. Let $I = [0,1]^2 \subseteq \mathbb{R}^2$ be the unit square. True or false:
   $$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^\gamma} \int_I R\Big(\frac{|x-y|}{\varepsilon}\Big) \, dx \, dy \quad \text{exists.}$$
   If yes, prove it. If no, find a counter example.
3. Let $\mu, \nu$ be two $\sigma$-finite positive measures on $(X, \Sigma)$. True or false: For every $\alpha > 0$ there exists $Y_\alpha \in \Sigma$ such that for every $A \in \Sigma$ we have $\nu(A \cap Y_\alpha) \leqslant \alpha \mu(A \cap Y_\alpha)$ and $\nu(A \cap Y_\alpha^c) \geqslant \alpha \mu(A \cap Y_\alpha^c)$. Prove it, or find a counter example.
4. Let $\mu$ be a $\sigma$-finite positive Borel measure on $\mathbb{R}^d$. Suppose there exists $c_0 \in [1, \infty)$ such that $\mu(B(x, 2r)) \leqslant c_0 \mu(B(x,r))$ for all $x \in \mathbb{R}^d$, $r > 0$. Let $f \in L^1(\mathbb{R}^d, \mu)$ define
   $$M_\mu f(x) = \sup_{r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f| \, d\mu$$
   Must $\displaystyle\sup_{\alpha>0} \alpha \mu(\{x \mid M_\mu f(x) > \alpha\}) < \infty$? If yes, prove it. If no, find a counter example.
5. Let $\mu$ be a finite measure on $(X, \Sigma)$, and $\nu$ be a finite measure on $(Y, \tau)$. Show that there exists a finite measure $\pi$ on the product $\sigma$-algebra $\Sigma \otimes \tau$ such that $\pi(A \times B) = \mu(A)\nu(B)$ for all $A \in \Sigma$, $B \in \tau$.
   Note: Even though this is a standard result in your syllabus, please provide a complete proof here. You may however use without proof standard lemmas that were proved independently of this fact in Fall 2020 720, or in standard reference listed on the Fall 2020 720 website.