Below, if not stated explicitly, \((X, \mathcal{F}, \mu)\) is a measure space, \(L_p = L_p(X, \mathcal{F}, \mu)\) is the standard \(L_p\) space \((p \in [1, \infty])\) and \(m\) is Lebesgue measure.

1. Find all positive real numbers \(\alpha\) and \(\beta\) such that
   \[
   I := \int_0^1 \int_0^1 \frac{1}{(x^\alpha + y)^\beta} \, dx \, dy < \infty.
   \]

2. Let \(f_n : X \to \mathbb{R}\) \((n = 1, 2, \ldots)\) be \(\mathcal{F}\)-measurable functions converging in measure to a function \(f\). Assume that there exists a \(\mu\)-integrable function \(g : X \to [0, \infty)\) such that
   \[|f_n(x)| \leq g(x) \quad \text{for } \mu\text{-a.e. } x \in X \text{ and all } n.\]
   Prove that \(\int_X |f_n - f| \, d\mu \to 0\) as \(n \to \infty\).

3. Let \(f : X \to [0, \infty)\) be measurable, and let \(g(t) = \mu(\{x : f(x) > t\})\) for all real \(t\). Prove (using elementary properties of the integral) that
   \[
   \int_X f^2 \, d\mu = \int_{[0,\infty)} 2t \, g(t) \, dm(t).
   \]

4. (a) Show that \(\bigcap_{p \in [1, \infty]} L_p \neq L_\infty\), in general.

   (b) If \(f \in \bigcap_{p \in [1, \infty]} L_p\), prove that the map \(p \mapsto p \log \|f\|_p\) is convex on \([1, \infty)\).

5. Let \(A \subset \mathbb{R}\) be Lebesgue measurable and have the property that \(A + r = A\) for all rational numbers \(r\). (Recall \(A + r = \{x + r : x \in A\}\).) If \(m(A) > 0\), prove that the complement of \(A\) has measure zero. (Hint: Consider integrals of \(1_A\) over intervals.)