Basic Examination: Measure and Integration September 2020

- This test is **closed book**: no notes or other aids are permitted.
- The exam has a total of 5 questions and 100 points (20 each).
- You have 3 hours. You must scan and email your solutions to the proctor within 15 minutes afterwards.
- You may use without proof *standard* results from the syllabus (not homework) which are independent of the question asked, unless explicitly instructed otherwise. You must, however, **clearly** state the result you are using.

Below, if not stated explicitly, $(X, \mathcal{F}, \mu)$ is a measure space, $L_p = L_p(X, \mathcal{F}, \mu)$ is the standard L_p space ($p \in [1, \infty]$) and m is Lebesgue measure.

1. Find all positive real numbers α and β such that

$$I := \int_0^1 \int_0^1 \frac{1}{(x^\alpha + y)^\beta}\, dx\, dy < \infty.$$

2. Let $f_n : X \to \mathbb{R}$ $(n = 1, 2, \ldots)$ be $\mathcal{F}$-measurable functions converging in measure to a function f. Assume that there exists a μ-integrable function $g : X \to [0, \infty)$ such that

$$|f_n(x)| \le g(x) \quad \text{for } \mu\text{-a.e. } x \in X \text{ and all } n.$$

Prove that $\displaystyle\int_X |f_n - f|\, d\mu \to 0$ as $n \to \infty$.

3. Let $f : X \to [0, \infty)$ be measurable, and let $g(t) = \mu(\{x : f(x) > t\})$ for all real t. Prove (using elementary properties of the integral) that

$$\int_X f^2\, d\mu = \int_{[0,\infty)} 2t\, g(t)\, dm(t).$$

4. (a) Show that $\bigcap_{p \in [1,\infty)} L_p \neq L_\infty$, in general.

(b) If $f \in \bigcap_{p \in [1,\infty)} L_p$, prove that the map $p \mapsto p \log \|f\|_p$ is *convex* on $[1, \infty)$.

5. Let $A \subset \mathbb{R}$ be Lebesgue measurable and have the property that $A + r = A$ for all rational numbers r. (Recall $A + r = \{x + r : x \in A\}$.) If $m(A) > 0$, prove that the complement of A has measure zero. (Hint: Consider integrals of $\mathbb{1}_A$ over intervals.)