Basic Examination: Measure and Integration January 2020

- This test is **closed book**: no notes or other aids are permitted.
- You have 3 hours. The exam has a total of 5 questions and 100 points (20 each).
- You may use without proof *standard* results from the syllabus which are independent of the question asked, unless explicitly instructed otherwise. You must, however, **clearly** state the result you are using.

Below, if not stated explicitly, $(X, \mathcal{F}, \mu)$ is a measure space, $L_p = L_p(X, \mathcal{F}, \mu)$ is the standard L_p space ($p \in [1, \infty]$) and m is Lebesgue measure.

1. Suppose $I = [0, 1]$, $f_n : I \to \mathbb{R}$ is Lebesgue measurable for all $n \in \mathbb{N}$, and

$$\int_{[0,1]} |f_n|^2 \, dm \le 5 \quad \text{for all } n \in \mathbb{N}.$$

Suppose moreover that $f_n(x) \to 0$ as $n \to \infty$, for every $x \in [0, 1]$.
(a) Does it necessarily follow that $\displaystyle\lim_{n\to\infty} \int_I |f_n|^2 \, dm = 0$?
(b) Does it necessarily follow that $\displaystyle\lim_{n\to\infty} \int_I |f_n| \, dm = 0$?
In each case, prove the implication or find a counterexample.

2. Let $f_n \in L_1$ for all $n \in \mathbb{N}$. Which is always larger,

$$\left(\sum_{n=1}^{\infty} \left| \int f_n \, d\mu \right|^2 \right)^{1/2} \qquad \text{or} \qquad \int \left(\sum_{n=1}^{\infty} |f_n|^2 \right)^{1/2} d\mu \quad ?$$

Prove your answer.

3. Suppose $\mu(X) < \infty$. Let $f : X \to \mathbb{R}$ be $\mathcal{F}$-measurable, and define

$$g(x, y) = f(x) - f(y), \qquad x, y \in X.$$

If g is $\mu \times \mu$-integrable on $X \times X$, is f necessarily μ-integrable? Prove or find a counterexample.

4. Let μ, ν, and ν_n be finite measures on $(X, \mathcal{F})$, and suppose that $\nu_n \ll \mu$ for all $n \in \mathbb{N}$. Is the following true or false?

$$\text{If } |\nu_n - \nu|(X) \to 0 \text{ as } n \to \infty \text{ then necessarily } \nu \ll \mu.$$

Prove or find a counterexample.

5. Let $K \subset \mathbb{R}^N$ be a compact set and let

$$E := \{x \in \mathbb{R}^N : \operatorname{dist}(x, K) = 1\}, \qquad F := \{x \in \mathbb{R}^N : \operatorname{dist}(x, K) < 1\}.$$

(a) Prove that for every $\varepsilon \in (0, \frac{1}{2})$ and every $x \in E$,

$$m(B(x_0, \varepsilon) \cap F) \ge \frac{1}{2^N} m(B(x_0, \varepsilon))$$

(b) Prove that E has Lebesgue measure zero.