DEPARTMENT OF MATHEMATICAL SCIENCES
CARNEGIE MELLON UNIVERSITY

BASIC EXAMINATION
MEASURE AND INTEGRATION
AUGUST 2019

Time allowed: 3 hours.

Name: ____

Problem	Points
1 (20)	
2 (20)	
3 (20)	
4 (20)	
5 (20)	
Total (100)	

1. (20 points) Let μ and ν be two positive measures on a measurable space $(X, \mathcal{M})$. Suppose that for every $\varepsilon > 0$ there exists E measurable such that $\mu(E) < \varepsilon$ and $\nu(E^c) < \varepsilon$. Show that $\mu \perp \nu$.

2. (20 points) Let $f \in L^\infty(\mathbb{R})$ be such that $f(x+1) = f(x)$ and $f(x) \geq 0$ for all $x \in \mathbb{R}$. Let $E \subset \mathbb{R}$ of finite Lebesgue measure. Show that

$$\lim_{n\to\infty} \int_E f(nx)dx = |E| \int_0^1 f(x)dx, \tag{1}$$

where $|E|$ is the Lebesgue measure of E.

3. (20 points) Let $(X, \mathcal{M}, \mu)$ be a measure space. Assume $0 < \mu(X) < \infty$.

   (i) Assume $f \in L^\infty(\mu)$. Prove that $f \in L^p(\mu)$ for all $p \in [1, \infty)$ and that

   $$\lim_{p \to \infty} \|f\|_p = \|f\|_\infty.$$

   (ii) Assume $f \in L^p(\mu)$ for all $p \in [1, \infty)$ and that there exists $M > 0$ such that $\|f\|_p \leq M$ for all $p \in [1, \infty)$. Show that $f \in L^\infty(\mu)$ and that $\|f\|_\infty \leq M$

4. (20 points) Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space. Let f be a measurable function such that $|f|^p \in L^1(\mu)$. Show that

(i) For $p \geq 1$, $\displaystyle\int |f|^p d\mu = p \int_0^\infty t^{p-1} \mu(\{x : |f(x)| > t\})\, dt.$

(ii) For $p > 1$, $\displaystyle\int |f|^p d\mu = (p-1) \int_0^\infty t^{p-2} \int_{\{x\,:\,|f(x)|>t\})} |f| d\mu\, dt.$

5. (20 points) Let $f_n : \mathbb{R} \to \mathbb{R}$ be a sequence of (Lebesgue) measurable functions such that

(a) $\{f_n\}_{n=1,2,\ldots}$ converges in measure to some function f.

(b) $|f_n(x)| \leq \frac{1}{x^2}$ for every $n \in N$ and $x \in \mathbb{R}$

(c) There exists $M \in \mathbb{R}$ such that $\|f_n\|_2 \leq M$ for all $n \in \mathbb{N}$.

For each statement below: Either prove the statement or provide a counterexample.

(i) $f_n \to f$ in $L^1(\mathbb{R})$ as $n \to \infty$.

(ii) $f_n \to f$ in $L^2(\mathbb{R})$ as $n \to \infty$.