**Basic Examination**

**Measure and Integration**

**August 2019**

Time allowed: 3 hours.

Name: ____________________________________________________________

<table>
<thead>
<tr>
<th>Problem</th>
<th>Points</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 (20)</td>
<td></td>
</tr>
<tr>
<td>2 (20)</td>
<td></td>
</tr>
<tr>
<td>3 (20)</td>
<td></td>
</tr>
<tr>
<td>4 (20)</td>
<td></td>
</tr>
<tr>
<td>5 (20)</td>
<td></td>
</tr>
<tr>
<td>Total (100)</td>
<td></td>
</tr>
</tbody>
</table>
1. (20 points) Let $\mu$ and $\nu$ be two positive measures on a measurable space $(X, \mathcal{M})$. Suppose that for every $\varepsilon > 0$ there exists $E$ measurable such that $\mu(E) < \varepsilon$ and $\nu(E^c) < \varepsilon$. Show that $\mu \perp \nu$. 
2. (20 points) Let $f \in L^\infty(\mathbb{R})$ be such that $f(x + 1) = f(x)$ and $f(x) \geq 0$ for all $x \in \mathbb{R}$. Let $E \subset \mathbb{R}$ of finite Lebesgue measure. Show that

$$\lim_{n \to \infty} \int_E f(nx) \, dx = |E| \int_0^1 f(x) \, dx,$$

where $|E|$ is the Lebesgue measure of $E$. 


3. (20 points) Let \((X, \mathcal{M}, \mu)\) be a measure space. Assume \(0 < \mu(X) < \infty\).

(i) Assume \(f \in L^\infty(\mu)\). Prove that \(f \in L^p(\mu)\) for all \(p \in [1, \infty)\) and that
\[
\lim_{p \to \infty} \|f\|_p = \|f\|_\infty.
\]

(ii) Assume \(f \in L^p(\mu)\) for all \(p \in [1, \infty)\) and that there exists \(M > 0\) such that \(\|f\|_p \leq M\) for all \(p \in [1, \infty)\). Show that \(f \in L^\infty(\mu)\) and that \(\|f\|_\infty \leq M\)
4. (20 points) Let \((X, \mathcal{M}, \mu)\) be a \(\sigma\)-finite measure space. Let \(f\) be a measurable function such that \(|f|^p \in L^1(\mu)\). Show that

(i) For \(p \geq 1\),
\[
\int |f|^p \, d\mu = p \int_0^\infty t^{p-1} \mu(\{x : |f(x)| > t\}) \, dt.
\]

(ii) For \(p > 1\),
\[
\int |f|^p \, d\mu = (p - 1) \int_0^\infty t^{p-2} \int_{\{x : |f(x)| > t\}} |f| \, d\mu \, dt.
\]
5. (20 points) Let \( f_n : \mathbb{R} \to \mathbb{R} \) be a sequence of (Lebesgue) measurable functions such that

(a) \( \{f_n\}_{n=1,2,...} \) converges in measure to some function \( f \).

(b) \( |f_n(x)| \leq \frac{1}{n^2} \) for every \( n \in \mathbb{N} \) and \( x \in \mathbb{R} \)

(c) There exists \( M \in \mathbb{R} \) such that \( \|f_n\|_2 \leq M \) for all \( n \in \mathbb{N} \).

For each statement below: Either prove the statement or provide a counterexample.

(i) \( f_n \to f \) in \( L^1(\mathbb{R}) \) as \( n \to \infty \).

(ii) \( f_n \to f \) in \( L^2(\mathbb{R}) \) as \( n \to \infty \).