DEPARTMENT OF MATHEMATICAL SCIENCES
CARNEGIE MELLON UNIVERSITY

BASIC EXAMINATION
MEASURE AND INTEGRATION
JANUARY 2019

Time allowed: 3 hours.

Do only 2 of the problems 4, 5, and 6.

Name: ____

Problem	Points
1 (20)	
2 (10)	
3 (30)	
4 (20)	
5 (20)	
6 (20)	
Total (100)	

1. (20 points) Let $f : \mathbb{R} \to \mathbb{R}$ be Lebesgue measurable. Show that there exists a Borel measurable function $g : \mathbb{R} \to \mathbb{R}$ and a set E of Lebesgue measure zero, $m(E) = 0$, such that

$$f(x) = g(x) \qquad \text{for all } x \in \mathbb{R} \setminus E.$$

Hint: Consider first simple functions.

2. (10 points) Let $\mathcal{L}^d$ be the Lebesgue measure on $\mathbb{R}^d$. Let $f \in L^1(\mathcal{L}^d)$ and let

$$g(t) = \mathcal{L}^d(\{x \,:\, |f(x)| > t\})$$

be the distribution function of $|f|$. Show that for any increasing differentiable function $\varphi : [0, \infty) \to [0, \infty)$ such that $\varphi(0) = 0$ it holds that

$$\int \varphi(|f(x)|) d\mathcal{L}^d = \int_0^\infty \varphi'(s) g(s) ds.$$

3. (30 points) Let $d \geq 1$, $f \in L^1(\mathbb{R}^d)$ and $g \in L^\infty(\mathbb{R}^d)$. Let $E \subset \mathbb{R}^d$ be a Lebesgue measurable set with positive measure, $\mathcal{L}^d(E) > 0$.

   (i) Show that $\lim_{h \to 0} \int |f(x+h) - f(x)| d\mathcal{L}^d = 0$.

   (ii) Show that

$$u(x) = f * g(x) = \int f(y)g(x-y)dy$$

   is a continuous function of $\mathbb{R}^d$.

   (iii) Show that the set $E - E = \{x - y \,:\, x, y \in E\}$ contains an open ball centered at the origin. that is show that there exists $r > 0$ such that $B(0, r) \subset E - E$.

4. (20 points) Let $(X, \mathcal{M}, \mu)$ be a measure space. Assume $f : X \to [0, \infty)$ is measurable and $\int f d\mu < \infty$. Show that

$$\lim_{n \to \infty} \int f^{\frac{1}{n}} d\mu = \mu\left(f^{-1}(0, \infty)\right).$$

5. (20 points) Let $f : \mathbb{R} \to \mathbb{R}$ be Borel measurable and integrable and let m be the Lebesgue measure on $\mathbb{R}$. Let

$$F(t) = \begin{cases} \int f \chi_{[0,t]} \, dm & \text{if } t \geq 0 \\ -\int f \chi_{[t,0]} \, dm & \text{if } t < 0. \end{cases}$$

Show that for any compactly supported smooth function, $g \in C_c^\infty(\mathbb{R})$

$$\int f g dm = -\int F g' dm.$$

6. (20 points) State and prove Caratheodory's Theorem.