Basic Examination: Measure and Integration—August 2018

• This test is closed book: no notes or other aids are permitted.

• You have 3 hours.

• You may use without proof standard results from the syllabus which are independent of the question asked, unless explicitly instructed otherwise. You must, however, clearly state the result you are using.

1. **(15 points)** State and prove Lusin's theorem.

2. Consider the integral
$$\int_0^\pi \int_x^\pi \frac{\sin y}{y} dy dx.$$
   (a) **(3 points)** Prove that the integral is well-defined.
   (b) **(7 points)** Determine the value of the integral.

3. Consider the Lebesgue measure $\mathcal{L}^1$ on the real line. Let $f : \mathbb{R} \to \mathbb{R}$ be an integrable function, let $\{a_n\}_n$ be a sequence of real numbers such that $a_n \to 0$ as $n \to \infty$, and for every $n \in \mathbb{N}$ let
$$f_n(x) := f(x + a_n), \quad x \in \mathbb{R}.$$
   (a) **(15 points)** Prove or give a counter example:
$$\lim_{n\to\infty} \int_{\mathbb{R}} |f_n - f| \, dx = 0.$$
   (b) **(15 points)** Prove or give a counter example. There exist a subsequence $\{f_{n_k}\}_k$ of $\{f_n\}_n$ and an integrable function $h : \mathbb{R} \to [0, \infty)$ such that
$$|f_{n_k}(x)| \leq h(x)$$
   for $\mathcal{L}^1$ a.e. $x \in \mathbb{R}$ and for all $k \in \mathbb{N}$.

4. **(15 points)** Let $\mathcal{L}^N$ be the Lebesgue measure in $\mathbb{R}^N$ and let $E \subset \mathbb{R}^N$ be a Borel set with the property that
$$\mathcal{L}^N(E \cap B) \leq \frac{1}{3}\mathcal{L}^N(B)$$
   for every ball $B \subset \mathbb{R}^N$. Prove that $\mathcal{L}^N(E) = 0$.

5. Consider the Lebesgue measure on the real line. Let $f \in L^p(\mathbb{R})$, $1 < p < \infty$, and let $\varphi \in C^1(\mathbb{R})$ have compact support and be such that $\int_{\mathbb{R}} \varphi(y) \, dy = 1$. Consider the function
$$g(x, y) = \frac{1}{x} \int_{\mathbb{R}} \varphi\left(\frac{y - t}{x}\right) f(t) \, dt, \quad (x, y) \in (0, \infty) \times \mathbb{R}.$$
   (a) **(10 points)** Prove that g is well-defined.

(b) **(10 points)** Prove that there exist $\frac{\partial g}{\partial x}$ and $\frac{\partial g}{\partial y}$ in $(0, \infty) \times \mathbb{R}$.

(c) **(10 points)** Assume that $f \in C_c(\mathbb{R})$. Given $y_0 \in \mathbb{R}$, does the limit $\lim_{(x,y)\to(0,y_0)} g(x, y)$ exist? Prove it, or find a counter example.