Basic Examination: Measure and Integration—August 2016

- This test is **closed book**: no notes or other aids are permitted.
- You have 3 hours. The exam has a total of 5 questions and 100 points (20 each).
- You may use without proof *standard* results from the syllabus which are independent of the question asked, unless explicitly instructed otherwise. You must, however, **clearly** state the result you are using.

Below, if not stated explicitly, $(X, \mathcal{F}, \mu)$ is a measure space, $L_p = L_p(X, \mathcal{F}, \mu)$ is the standard L_p space ($p \in [1, \infty]$) and $C_c(X)$ is the space of continuous functions from X to $\mathbb{R}$ having compact support. Moreover, m is Lebesgue measure, $\mathcal{L}$ is the σ-algebra of Lebesgue-measurable sets, and $\mathcal{B}(X)$ is the Borel σ-algebra of subsets of X.

1. Prove or find a counterexample: If $f\colon (0,1) \to (-\infty, \infty)$ is continuous and $\lim_{\varepsilon \to 0^+} \int_\varepsilon^1 f\, dm$ exists and is finite, then f is Lebesgue integrable on $(0,1)$.

2. Suppose $f_n\colon X \to [0, \infty)$ is measurable for each $n \in \mathbb{N}$ and $\mu(X) < \infty$. Show that the sequence $\{f_n\}$ converges to 0 in measure if and only if

$$\int_X \frac{f_n}{1 + f_n}\, d\mu \to 0 \quad \text{as } n \to \infty.$$

3. Suppose that for each $n \in \mathbb{N}$, $f_n = \mathbb{1}_{E_n}$ for some Lebesgue measurable set $E_n \subset [0,1]^d$, and

$$\mu(A) \stackrel{\text{def}}{=} \lim_{n \to \infty} \int_A f_n\, dm$$

exists and is finite for each Borel set $A \subset [0,1]^d$. Show that μ is a Borel measure, and μ is absolutely continuous with respect to Lebesgue measure m on $[0,1]^d$, with Radon-Nikodym derivative $d\mu/dm$ taking values in $[0,1]$ a.e.

4. Suppose f and g are positive and measurable on $X = [0,1]$, and satisfy

$$f(x)g(x) \geq 1 \quad \text{for all } x \in [0,1]. \tag{1}$$

(i) Show that $\left(\int_X f\, dm\right)\left(\int_X g\, dm\right) \geq 1.$

(ii) For which choices of $p, q \in (0, \infty)$ does the assumption (1) imply

$$\left(\int_X f^p\, dm\right)^{1/p} \left(\int_X g^q\, dm\right)^{1/q} \geq 1 \ \ ?$$

5. Let $u\colon \mathbb{R} \to (0, \infty)$ be Lebesgue measurable and 1-periodic. Let $X = [0,1]$ and suppose $\int_X u\, dm = 1$. For each $\varepsilon > 0$, define $u_\varepsilon(x) = u(x/\varepsilon)$, and define a measure μ_ε on $\mathcal{B}(X)$ by

$$\mu_\varepsilon(A) \stackrel{\text{def}}{=} \int_A u_\varepsilon\, dm\,.$$

(i) Prove that $\mu_\varepsilon(A) \to m(A)$ as $\varepsilon \to 0^+$ for every $A \in \mathcal{B}(X)$.

(ii) Let $\nu_\varepsilon \stackrel{\text{def}}{=} |\mu_\varepsilon - m|$ denote the total variation of the signed measure $\mu_\varepsilon - m$ on X. Show that if $\nu_\varepsilon(X) \to 0$ as $\varepsilon \to 0^+$, then $u(x) = 1$ a.e.