Basic Qualification Exam: Measure Theory.

Aug 31, 2015

- *This is a closed book test. No calculators or computational aids are allowed.*
- *You have 2 hours. The exam has a total of 4 questions and 20 points.*
- *You may use without proof* standard *results from the syllabus which are independent of the question asked,* unless *explicitly instructed otherwise. You must, however,* ***CLEARLY*** *state the result you are using.*

Unless otherwise stated, we always assume the underlying measure space is (X, Σ, μ) and μ is a positive measure. The Lebesgue measure on $\mathbb{R}^d$ will be denoted by λ.

$\boxed{5}$ 1. Find all $\alpha \in [0, \infty]$ for which there exists a Lebesgue measurable function $f : \mathbb{R} \to [0, \infty)$ such that

$$\lim_{n \to +\infty} \int_{[n,\infty)} f \, d\lambda = \alpha.$$

$\boxed{5}$ 2. Let $1 \leqslant p < q < r \leqslant \infty$. True or false:

If $g \in L^q(X)$, then there exists $f \in L^p(X)$ and $h \in L^r(X)$ such that $g = f + h$.

Prove it, or find a counter example.

$\boxed{5}$ 3. Prove the following special case of the fundamental theorem of Calculus. If $f : [0, 1] \to \mathbb{R}$ is absolutely continuous, then show that f is differentiable almost everywhere, $f' \in L^1([0, 1])$ and $f(1) - f(0) = \int_0^1 f'$.

$\boxed{5}$ 4. True or false:

If $f, g \in L^1(\mathbb{R}^2)$, then there exists $y \in \mathbb{R}^2$ such that $\displaystyle\int_{\mathbb{R}^2} \big|f(x - y)g(x + y)\big| \, d\lambda(x) < \infty$

Prove it, or find a counter example.