Basic Qualification Exam: Measure Theory.

Jan 12, 2015

- *This is a closed book test. No calculators or computational aids are allowed.*
- *You have 2 hours. The exam has a total of 4 questions and 20 points.*
- *You may use without proof* standard *results from the syllabus which are independent of the question asked,* unless *explicitly instructed otherwise. You must, however,* ***CLEARLY*** *state the result you are using.*

Unless otherwise stated, we always assume the underlying measure space is (X, Σ, μ) and μ is a positive measure. The Lebesgue measure on $\mathbb{R}^d$ will be denoted by λ.

[5] 1. True or false:

If $A, B \subseteq \mathbb{R}^d$ are such that $A \subset \{x_1 \geqslant 0\}$ and $B \subset \{x_1 < 0\}$, then $\lambda^*(A \cup B) = \lambda^*(A) + \lambda^*(B)$.

Prove it, or find a counter example. [Note, A and B are not assumed to be Lebesgue measurable. Above λ^* denotes the Lebesgue outer measure on $\mathbb{R}^d$.]

[5] 2. Assume $f_n : \mathbb{R} \to \mathbb{R}$ is a sequence of functions such that (f_n) converges in measure, $|f_n(x)| \leqslant 1/x^2$ for every $n \in \mathbb{N}$ and $x \in \mathbb{R}$, and $\sup_n \int_{\mathbb{R}} |f_n|^2 \, d\lambda < \infty$.

(a) True or false: the above assumptions imply that the sequence (f_n) is also convergent in $L^1(\mathbb{R})$. Prove it, or find a counter example.

(b) True or false: the above assumptions imply that the sequence (f_n) is also convergent in $L^2(\mathbb{R})$. Prove it, or find a counter example.

[5] 3. True or false:

If $f : \mathbb{R}^2 \to \mathbb{R}$ is Lipschitz, then $\partial_1 f(x)$ exists for almost every x in $\mathbb{R}^2$.

Prove it, or find a counter example. [Recall a function f is said to be Lipschitz if there exists a constant C such that for all x, y in the domain of f we have $|f(x) - f(y)| \leqslant C|x - y|$.]

[5] 4. True or false:

Let $U, V \subseteq \mathbb{R}^2$ be open, and $\varphi : U \to V$ be Lipschitz. If $K \subset U$ is compact and $\lambda(K) = 0$, then $\varphi(K)$ is Lebesgue measurable and $\lambda(\varphi(K)) = 0$.

Prove it, or find a counter example.