Basic Qualification Exam: Measure Theory.

Aug 28, 2014

- *This is a closed book test. No calculators or computational aids are allowed.*
- *You have 2 hours. The exam has a total of 4 questions and 20 points.*
- *You may use without proof* standard *results from the syllabus which are independent of the question asked,* unless *explicitly instructed otherwise. You must, however,* ***CLEARLY*** *state the result you are using.*

Unless otherwise stated, we always assume the underlying measure space is (X, Σ, μ) and μ is a positive measure. The Lebesgue measure on $\mathbb{R}^d$ will be denoted by λ.

$\boxed{5}$ 1. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be Borel measurable, and $g : \mathbb{R}^2 \to \mathbb{R}^2$ be Lebesgue measurable.

(a) True or false: $f \circ g$ is necessarily Lebesgue measurable. (No proof required.)

(b) True or false: $g \circ f$ is necessarily Lebesgue measurable. Prove it, or find a counter example.

$\boxed{5}$ 2. Define $R \subset \mathbb{R}^2$ be the rectangle defined by $R = [1, \infty] \times [-1, 1]$. For each of the functions below determine if the Lebesgue integral $\int_R f \, d\lambda$ exists in the extended sense. If yes, compute it. Justify your answer.

(a) $f(x, y) = \sin(xy) \exp(-x^2 y^2)$.

(b) $f(x, y) = \sin(xy) \exp(-x^2 + y^2)$.

$\boxed{5}$ 3. Prove the following special case of the fundamental theorem of calculus.

If $f : [0, 1] \to \mathbb{R}$ is absolutely continuous and strictly increasing, then f is differentiable almost everywhere, $f' \in L^1$ and $f(1) - f(0) = \displaystyle\int_0^1 f' \, d\lambda$.

[You may not use the fundamental theorem of calculus for this part. You may, however, other results that were proved independently of this fact in either 720 or a measure theory reference listed on the 2013 720 website.]

$\boxed{5}$ 4. Let $P \subset [1, \infty]$ be the set of all $p \in [1, \infty]$ for which the following statement true:

If $f_n, f : [0, 1] \to \mathbb{R}$ are integrable functions such that $\sup_{n \in \mathbb{N}} \|f_n\|_{L^p} < \infty$ and $\displaystyle\lim_{n \to \infty} \int_A f_n \, d\lambda = \int_A f \, d\lambda$ for every $A \in \mathcal{B}([0, 1])$, then $(f_n) \to f$ in measure.

Find P. Prove the above statement for all $p \in P$, and provide a counter example for all $p \in P^c \cap [1, \infty]$.