Department of Mathematical Sciences
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Basic Examination
Measure and Integration
January 2012

Time allowed: 120 minutes.
Do four of the five problems. Indicate on the first page which problems you have chosen to be graded. All problems carry the same weight.

1. Let X be a nonempty set, let $\mathfrak{M}$ be an algebra on X, and let $f : X \to [0,\infty]$ be a measurable function. Prove that there exists a sequence $\{s_n\}$ of simple functions such that
$$0 \le s_1(x) \le s_2(x) \le \ldots \le s_n(x) \to f(x)$$
for every $x \in X$ and that the convergence is uniform on any set on which f is bounded from above.

2. Let $(X, \mathfrak{M}, \mu)$ be a measure space and let $1 \le p < \infty$. Prove that $L^p(X)$ is a complete metric space.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a Borel function, integrable on compact sets, and for every $\varepsilon > 0$, let
$$f_\varepsilon(x) := \frac{1}{2\varepsilon}\int_{x-\varepsilon}^{x+\varepsilon} f(t)\, dt.$$
   (a) Prove that
$$\int_{\mathbb{R}} |f_\varepsilon(x)|\, dx \le \int_{\mathbb{R}} |f(x)|\, dx.$$
   (b) Prove that if $f \in C_c(\mathbb{R})$, then $f_\varepsilon(x) \to f(x)$ as $\varepsilon \to 0^+$ for every $x \in \mathbb{R}$.
   (c) Prove that if f is integrable, then $f_\varepsilon(x) \to f(x)$ as $\varepsilon \to 0^+$ for $\mathcal{L}^1$ a.e. $x \in \mathbb{R}$.

4. Given the function $f : [0,1] \times [0,1] \to \mathbb{R}$ defined by
$$f(x,y) = \begin{cases} \dfrac{1}{\left(x - \frac{1}{2}\right)^3} & \text{if } 0 < y < \left|x - \frac{1}{2}\right|, \\ 0 & \text{otherwise,} \end{cases}$$
determine if the integrals
$$\int_0^1 \left(\int_0^1 f(x,y)\, dx\right) dy, \quad \int_0^1 \left(\int_0^1 f(x,y)\, dy\right) dx, \quad \int_{[0,1]\times[0,1]} f(x,y)\, d\mathcal{L}^2$$
are well-defined and, in case, calculate their value.

5. Let $(X, \mathfrak{M}, \mu)$ be a measure space and let $\mu_n : \mathfrak{M} \to [0,\infty]$ be measures.

(a) Prove that the set function $\bigvee_{n=1}^{\infty} \mu_n : \mathfrak{M} \to [0, \infty]$, defined by

$$\bigvee_{n=1}^{\infty} \mu_n (E) := \sup \left\{ \sum_{n=1}^{\infty} \mu_n (E_n) : \, E_n \in \mathfrak{M}, \, E_n \text{ pairwise disjoint}, \, \bigcup_{n=1}^{\infty} E_n = E \right\}, \quad E \in \mathfrak{M},$$

is a measure.

(b) Prove that if $\nu : \mathfrak{M} \to [0, \infty]$ is a measure such that $\mu_n \leq \nu$ for every n, then $\bigvee_{n=1}^{\infty} \mu_n \leq \nu$.

(c) Prove that if μ is finite and each μ_n is absolutely continuous with respect to μ, then

$$\bigvee_{n=1}^{\infty} \mu_n (E) = \int_E \left(\sup_n f_n \right) d\mu$$

for every $E \in \mathfrak{M}$, where f_n is the Radon-Nikodym derivative of μ_n with respect to μ.