Department of Mathematical Sciences
Carnegie Mellon University

Basic Examination
Measure and Integration
August 2011

Time allowed: 120 minutes.
Do four of the five problems. Indicate on the first page which problems you have chosen to be graded. All problems carry the same weight.

1. State and prove Vitali's convergence theorem.

2. State and prove Dynkin's theorem.

3. Let
$$E := \left\{ (x,y,z) \in \mathbb{R}^3 : \, z \geq 1, \;\; |x| + |y| < \frac{1}{\sqrt{z}} \right\},$$
$$f(x,y,z) := z^{1/4}(x+y).$$
   (a) Prove that E is a Borel set and compute its Lebesgue measure $\mathcal{L}^3(E)$.
   (b) Prove that the Lebesgue integral $\int_E f \, d\mathcal{L}^3$ is well-defined and calculate its value.

4. Let $f_n : \mathbb{R} \to \mathbb{R}$ be continuous functions, $n \in \mathbb{N}$.
   (a) Prove that the set
   $$F := \left\{ x \in \mathbb{R} : \lim_{n \to \infty} f_n(x) = \infty \right\}$$
   is a Borel set.
   (b) Prove that the set
   $$E := \{ x \in \mathbb{R} : \, f_n(x) \in [0,1] \text{ for infinitely many } n \}$$
   is a Borel set.

5. Let $(X, \mathfrak{M}, \mu)$ be a measure space and let $\mathfrak{N} \subset \mathfrak{M}$ be a family closed under finite unions and such that $\emptyset \in \mathfrak{N}$. Prove that the set function $\mu_1 : \mathfrak{M} \to [0, \infty]$, defined by
$$\mu_1(E) := \sup \{ \mu(E \cap F) : \, F \in \mathfrak{N} \}, \quad E \in \mathfrak{M}, \tag{1}$$
is a measure.