Department of Mathematical Sciences
Carnegie Mellon University

Basic Examination
Measure and Integration
January 2011

Time allowed: 120 minutes.
Do four of the five problems. Indicate on the first page which problems you have chosen to be graded. All problems carry the same weight.

1. State and prove Lebesgue's monotone convergence theorem.

2. Let $(X, \mathfrak{M}, \mu)$ be a finite measure space and let f_n, $f : X \to \mathbb{R}$ be measurable functions such that $\{f_n\}$ converges to f in measure. Prove that if μ is finite and $g : \mathbb{R} \to \mathbb{R}$ is a continuous function, then $\{g \circ f_n\}$ converges to $g \circ f$ in measure.

3. Let $(X, \mathfrak{M}, \mu)$ be a measure space, let $1 < p < \infty$ and let $f, g \in L^p(X)$. Prove that the function
$$h(t) := \int_X |f + tg|^p \, d\mu, \quad t \in \mathbb{R},$$
is differentiable at $t = 0$ and find $h'(0)$. What happens for $p = 1$?

4. Let $\nu : \mathcal{B}(0, \infty) \to [0, \infty]$ be a measure finite on compact sets and let μ be the Lebesgue measure restricted to $\mathcal{B}(0, \infty)$. Assume that $\nu \ll \mu$, that $\nu(B) = \nu(aB)$ for every Borel set $B \subset (0, \infty)$ and for every $a > 0$, and that $\frac{d\nu}{d\mu}$ is a continuous function. Prove that $\frac{d\nu}{d\mu}(x) = \frac{c}{x}$ for some constant $c \geq 0$ and all $x > 0$.

5. Let $\mu : \mathcal{B}(\mathbb{R}^N) \to [0, \infty]$ be a measure finite on compact sets, let $1 \leq p < \infty$ and let $f : \mathbb{R}^N \to \mathbb{R}$ be such that $[f] \in L^p(\mathbb{R}^N)$. Prove that there exists a Borel set $E \subset \mathbb{R}^N$, with $\mu(E) = 0$, such that for every $x \in \mathbb{R}^N \setminus E$,
$$\lim_{r \to 0^+} \frac{1}{\mu\left(\overline{B(x, r)}\right)} \int_{\overline{B(x,r)}} |f(y) - f(x)|^p \, d\mu(y) = 0.$$