Department of Mathematical Sciences
Carnegie Mellon University

Basic Examination
Measure and Integration
August 2010

Time allowed: 120 minutes.
Do four of the five problems. Indicate on the first page which problems you have chosen to be graded. All problems carry the same weight.

1. Let $(X, \mathfrak{M}, \mu)$ be a measure space and let f_n, $f : X \to \mathbb{R}$ be measurable functions. Prove that if $\{f_n\}$ converges to f in measure, then there exists a subsequence $\{f_{n_k}\}$ converging to f pointwise almost everywhere.

2. Let $p > 1$ and let $f_n : [0,1] \to \mathbb{R}$, $n \in \mathbb{N}$, be Lebesgue measurable functions such that
$$f_n(x) \to f(x)$$
for all $x \in [0,1]$ for some function $f : [0,1] \to \mathbb{R}$ and
$$\int_0^1 |f_n(x)|^p \, dx \le C \quad \text{for all } n \in \mathbb{N},$$
for some $p > 1$ and for some constant $C > 0$. Prove that $f \in L^p([0,1])$, that $f_n \to f$ in $L^q([0,1])$ for all $1 < q < p$, but that in general $\{f_n\}$ needs not converge to f in $L^p([0,1])$ (give a counterexample).

3. Let X be a nonempty set, let $\mathfrak{M} \subset \mathcal{P}(X)$ be an *algebra, and let* $\mu : \mathfrak{M} \to [0, \infty]$ be a finitely additive measure. Prive that μ is countably additive if and only if
$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} \mu(E_n)$$
for every increasing sequence $\{E_n\} \subset \mathfrak{M}$ such that $\bigcup_{n=1}^{\infty} E_n \in \mathfrak{M}$.

4. Consider the function
$$f(x) = \frac{\sin x}{x^a}, \quad x > 0,$$
where $a > 0$. Determine for which values of the parameter $a > 0$ the function f is Riemann integrable (in the sense of improper integrals) over $(0, \infty)$ and for which it is Lebesgue integrable over $(0, \infty)$.

5. Given the sequence of functions
$$f_n(x) = \frac{e^{-n(x-n)}}{1+x^2} \chi_{[n,\infty)}(x),$$
determine the largest subset of $\mathbb{R}$ where the sequence converges pointwise, and if the convergence on the set is uniform. Calculate the limit
$$\lim_{n\to\infty} \int_0^{\infty} f_n(x) \, dx.$$