BASIC EXAMINATION SAMPLE
# GENERAL TOPOLOGY

**Do four of the five problems.**

1. Consider the topology on $\mathbb{R}$ for which $\mathcal{A} = \{[a,b) \ : \ a,b \in \mathbb{R}\}$ is a subbasis.

   (a) Prove that for all $x \in \mathbb{R}$ there exists a countable local basis.

   (b) Show that the space does not have a countable basis of topology.

2. Show that every compact Hausdorff topological space is normal.

3. Let $f : [a,b] \to \mathbb{R}$ and let

$$\operatorname{gr} f := \{(x, f(x)) : \ x \in [a,b]\}$$

   be the graph of $f$.

   Prove that the following two conditions are equivalent:

   (i) $f$ is continuous.

   (ii) $\operatorname{gr} f$ is compact.

4. Let $f_n : [0,1] \to [0,1]$ be a sequence of functions such that for all $n$ and all $x, y \in [0,1]$ such that $|x-y| > 1/n$

$$|f_n(x) - f_n(y)| \leq \frac{1}{n}|x-y|.$$

   Show that $f_n$ has a uniformly convergent subsequence.

5. Let $(X,d)$ be a complete metric space and $S : X \to X$ such that $S^2 := S \circ S$ is a strict contraction. That is, there exists $\alpha \in [0,1)$ such that

$$d(S^2(x), S^2(y)) \leq \alpha d(x,y) \quad \text{for all } x, y \in X.$$

   Show that the mapping $S$ has exactly one fixed point.