BASIC EXAMINATION SAMPLE
GENERAL TOPOLOGY

Do four of the five problems.

1. Let $a \in \mathbb{R}$ and consider the sequence of functions

$$f_n(x) = n^a x e^{-nx}, \quad x \in \mathbb{R}.$$

   (a) Find the largest set $E \subset \mathbb{R}$ where there is pointwise convergence to some function $f : E \to \mathbb{R}$.

   (b) Is the convergence uniform on E? (Prove your answer.)

2. State and prove Urysohn's lemma.

3. Consider the metric space $(C([0,1],\mathbb{R}), d)$, where

$$d(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)|$$

   Let $f \in C([0,1],\mathbb{R})$. Show that $B(f,1)$ is not compact.

4. Let $K \in C([-1,1],\mathbb{R})$. Consider the set of continuous functions $X = C([0,1],\mathbb{R})$. Given $f \in X$ let $Tf : [0,1] \to \mathbb{R}$ be defined by

$$Tf(x) := \int_0^1 K(x-y)f(y)dy.$$

   Let

$$\mathcal{F} := \{Tf \ : \ f \in X \text{ and } \max_{x\in[0,1]} |f(x)| \le 1\}$$

   Show that every sequence in $\mathcal{F}$ has a uniformly convergent subsequence.

5. Given a set X, consider the infinite product $[0,1]^X$ of all functions $f : X \to [0,1]$.

   (a) Prove that $[0,1]^{[0,1]}$ is not sequentially compact with the product topology (but it is compact by Tychonoff's theorem).

   (b) Prove that $[0,1]^{\mathbb{N}}$ is not compact with the box topology.