

GENERAL TOPOLOGY BASIC EXAM: JANUARY 2025

Hand in solutions to *exactly five* of the following questions. All questions carry equal weight.

- (1) Let X be a space such that arbitrary intersections of open sets are open. Show that X either is not T_1 or it is a discrete space.
- (2) Define the following concepts:
 - A *net* in a topological space.
 - *Convergence* of a net to a point.
 - A *subnet* of a net.
 - An *ultranet*.
 Prove that for every space X :
 - (a) X is Hausdorff if and only if every net in X converges to at most one point.
 - (b) X is compact if and only if every net in X has a convergent subnet.
- (3) For $m, n \in \mathbb{Z}$ let $d(m, n)$ be defined as follows:

$$d(m, n) = \begin{cases} 0 & \text{if } m = n \\ 2^{-t} & t \text{ is the largest integer such that } 2^t \text{ divides } m - n \text{ if } m \neq n \end{cases}$$

- (a) Prove that d is a metric on $\mathbb{Z}$.
 - (b) Let X be the completion of $\mathbb{Z}$ with respect to the metric d .
 - (i) Prove that X is compact.
 - (ii) Determine whether X is connected.
- (4) Define the concept of a *normal space*. Prove that:
 - (a) Every compact Hausdorff space is normal.
 - (b) Every metric space is normal.
 - (c) If C and D are disjoint closed subsets of a normal space X , then there is a continuous $f: X \rightarrow [0, 1]$ such that $f(c) = 0$ for all $c \in C$ and $f(d) = 1$ for all $d \in D$. It will be enough for full credit to describe the family of sets that is constructed in the course of the proof, the way in which the function f is defined from that family, and the argument that f is continuous.
- (5) Let E and B be spaces and let $p: E \rightarrow B$ be a *covering map*: that is to say for every $b \in B$ there is an open neighbourhood U of b , such that $p^{-1}[U]$ is the union of a family $(V_i)_{i \in I}$ of pairwise disjoint open sets such that $p \upharpoonright V_i$ is a homeomorphism between V_i and U for all i .

Prove that if B is connected, $p: E \rightarrow B$ is a covering map, and there exist a point $b_0 \in B$ and an integer k such that $|\{e \in E : p(e) = b_0\}| = k$, then $|\{e \in E : p(e) = b\}| = k$ for all $b \in B$.
- (6) State the Baire category theorem. Use it to prove that there is a continuous real valued function from $[0, 1]$ to $\mathbb{R}$ which is neither increasing or decreasing on any interval of positive length contained in $[0, 1]$. Hint: You may assume without proof that the continuous functions from $[0, 1]$ to $\mathbb{R}$ form a complete metric space under the metric $d(f, g) = \sup_{x \in [0, 1]} |f(x) - g(x)|$.