General Topology: Basic Exam

August 28, 2024

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Name: ______________________________

Problem	Points
1	
2	
3	
4	
5	
Total	

Each problem is worth 20 points.

Problem 1 **20 points**

Let X and Y be topological spaces, where Y is connected. Let $p\colon X \to Y$ be a quotient map such that for every $y \in Y$ the preimage $p^{-1}(\{y\})$ is connected. Show that X is connected.

Problem 2 **20 points**

Denote by ℓ^∞ the space of bounded sequences in $\mathbb{R}$ with the metric

$$d_\infty\colon \ell^\infty \times \ell^\infty \to [0,\infty), \quad d_\infty((x_n)_n, (y_n)_n) = \sup_n |x_n - y_n|.$$

Let (X,d) be a separable metric space. Show that X isometrically embeds into ℓ^∞, that is, show that there is a map $f\colon X \to \ell^\infty$ with $d_\infty(f(x), f(y)) = d(x,y)$ for all $x, y \in X$.

Recall that a space X is separable if there is a countable set $D \subset X$ that is dense in X.

Hint: It might be helpful to first solve this for (X,d) a bounded, separable metric space and then adjust the solution for the general case.

Problem 3 **20 points**

Let (X,τ) be a topological space such that every compact $C \subset X$ is closed. Then for any strictly finer topology $\tau' \supset \tau$, the space (X,τ') is not compact.

Problem 4 **20 points**

Let X be a topological space, and let Y and Z be two compact Hausdorff spaces. Suppose there are embeddings $f\colon X \to Y$ and $g\colon X \to Z$ such that $f(X)$ is dense in Y, $g(X)$ is dense in Z, $|Y \setminus f(X)| = 1$, and $|Z \setminus g(X)| = 1$. Show that Y and Z are homeomorphic.

Problem 5 **20 points**

As usual, let $S^1 = \{(x_1,x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 1\} \subset \mathbb{R}^2$ denote the circle. Let $f\colon S^1 \to S^1$ be any continuous function with $f(x) \neq x$ for all $x \in S^1$. Show that $g\colon S^1 \to S^1$ defined by $g(x) = \frac{f(x)-x}{|f(x)-x|}$ is not homotopic to a constant map.