General Topology: Basic Exam

January 17, 2024

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Name: ______________________________

Problem	Points
1	
2	
3	
4	
5	
Total	

Each problem is worth 20 points.

Problem 1 **20 points**

Let X be a connected and locally path-connected space. Show that X is path-connected.

Hint: Fix a point $x_0 \in X$ and show that the set of points that can be reached from x_0 by a path is both open and closed.

Problem 2 **20 points**

Let X be a normal space that contains a closed subset $C \subseteq X$ such that C is homeomorphic to $[0, \infty)$. Show that there is a continuous map $f\colon X \to X$ that has no fixed point.

Problem 3 **20 points**

Let X be a first countable space. Suppose there is a sequence $K_1, K_2, \ldots$ of compact subsets of X such that for every compact set $C \subseteq X$ there is an $n \in \mathbb{N}$ such that $C \subseteq K_n$. Show that every point $x \in X$ has a compact neighbourhood.

Problem 4 **20 points**

Let X be a discrete space with Stone–Čech compactification $\Phi\colon X \to \beta X$. Show the following:

(a) For $A \subseteq X$ the closure of $\Phi(A)$ in βX is disjoint from the closure of $\Phi(X \setminus A)$ in βX.
(b) If $U \subseteq \beta X$ is open, then the closure $\overline{U}$ is open in βX.

Problem 5 **20 points**

Show that $f\colon X \to Y$ is a homotopy equivalence if there exist maps $g, h\colon Y \to X$ such that $f \circ g$ is homotopic to the identity on Y and $h \circ f$ is homotopic to the identity on X.