General Topology: Basic Exam

August 31, 2023

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Name: ______________________________

Problem	Points
1	
2	
3	
4	
5	
Total	

Each problem is worth 20 points.

Problem 1 **20 points**

(a) Let X be a compact, locally connected topological space. Show that X has finitely many connected components.

(b) Show that $\{0,1\}^{\mathbb{N}}$ is not locally connected. Here $\{0,1\}$ carries the discrete topology, and $\{0,1\}^{\mathbb{N}}$ the product topology.

Problem 2 **20 points**

A space X is *perfectly normal* if X is normal and every closed set in X is a countable intersection of open sets. Show that X is perfectly normal if and only if for every closed $C \subseteq X$ there is a continuous $f\colon X \to [0,1]$ with $f^{-1}(0) = C$.

Hint: If $f_n\colon X \to [0,1]$ is a sequence of functions, then $\sum_n \frac{1}{2^{n+1}} f_n(x)$ defines a function $X \to [0,1]$.

Problem 3 **20 points**

Let $\mathbb{R}^{\mathbb{N}}$ be the space of real-valued sequences with the product topology, where $\mathbb{R}$ has the standard topology. Show that the subset

$$B = \{x \in \mathbb{R}^{\mathbb{N}} \;:\; \sup_{n\in\mathbb{N}} |x_n| < \infty\}$$

of bounded sequences is dense in $\mathbb{R}^{\mathbb{N}}$ and has empty interior.

Problem 4 **20 points**

Let X be a metrizable space. Show that the following statements are equivalent:

(i) X is bounded under every metric that induces the given topology on X.

(ii) Every continuous function $f\colon X \to \mathbb{R}$ is bounded.

(iii) X is compact.

Hint: For (i) $\Rightarrow$ (ii) given $f\colon X \to [0,1]$, consider the graph of f, which is homeomorphic to X.

Problem 5 **20 points**

Let X be a compact metric space. Let $\{f_n\colon X \to X \;:\; n \in \mathbb{N}\}$ be a set of functions that are equicontinuous at every point of X. Suppose that the image $f_n(X)$ is homeomorphic to the closed ball $B^k = \{x \in \mathbb{R}^k \;:\; |x| \leq 1\}$ of possibly varying dimensions k for each n. Suppose further that the sequence $(f_n)_n$ converges pointwise to the function $f\colon X \to X$. Show that f has a fixed point.