# General Topology: Basic Exam

January 25, 2023

*Do not flip the page until instructed.*

Name: ______________________________

| Problem | Points |
|---|---|
| 1 | |
| 2 | |
| 3 | |
| 4 | |
| 5 | |
| Total | |

Each problem is worth 20 points.

**Problem 1** **20 points**

Recall Urysohn's lemma: If $X$ is a normal space and $A, B \subseteq X$ are closed, nonempty, and disjoint subsets of $X$, then there is a continuous map $f\colon X \to [0,1]$ with $f(A) = \{0\}$ and $f(B) = \{1\}$.

Let $X$ be a normal space that is countable. Show that in $X$ every path-component consists of a single point. In fact, show that every connected component of $X$ consists of a single point.

**Problem 2** **20 points**

(a) State Tietze's extension theorem.

(b) Let $J$ be a set. Let $X$ be a normal space, $A \subseteq X$ a closed subspace, and $f\colon A \to \mathbb{R}^J$ a continuous map. Show that there is a continuous map $F\colon X \to \mathbb{R}^J$ with $F|_A = f$.

(c) A normal space $Y$ is an *absolute retract* if for any normal $Z$ and closed subspace $Y_0 \subseteq Z$ homeomorphic to $Y$, there is a continuous map $r\colon Z \to Y_0$ with $r(y) = y$ for all $y \in Y_0$. Let $X$ be a normal space. Show that if $Y$ is an absolute retract and compact, then for any closed $A \subseteq X$ and $f\colon A \to Y$ a continuous map, there is a continuous map $F\colon X \to Y$ with $F|_A = f$.

**Problem 3** **20 points**

(a) Give the definition of a locally path-connected space.

(b) Let $X$ be a Hausdorff space, and let $f\colon [0,1] \to X$ be continuous and surjective. Show that $X$ is locally path-connected.

**Problem 4** **20 points**

Let $X$ be a non-compact space, and let $\Phi\colon X \to K$ be a metrizable compactification of $X$, that is, $\Phi$ is an embedding such that $\Phi(X)$ is dense in $K$, and $K$ is metrizable and compact. Show that there is a metrizable compactification $\Psi\colon X \to K'$ of $X$ such that $K$ is isometric to a subspace of $K'$, but $K$ is not isometric to $K'$.

*Hint:* First construct a continuous map $f\colon X \to [0,1]$ that does not extend to a continuous map $\widehat{f}\colon K \to [0,1]$. Now use $f$ and $K$ to construct $K'$.

**Problem 5** **20 points**

(a) State Brouwer's fixed point theorem.

(b) Let $x_0 \in S^1$ be some point on the circle $S^1$. Let $X = \{(x,y) \in S^1 \times S^1 \ : \ x = x_0 \text{ or } y = x_0\}$. Show that there is a quotient map $q\colon [0,1]^2 \to S^1 \times S^1$ such that $q^{-1}(X) = \partial[0,1]^2$, the boundary of the square $[0,1]^2$.

(c) Let $f\colon S^1 \times S^1 \to S^1 \times S^1$ be continuous. Show that if $f$ has no fixed points, then there is an $x \in S^1 \times S^1$ with $f(x) \in X$.