Department of Mathematical Sciences
Basic Exam in General Topology
August 2022

Instructions: Answer each of the 5 questions.

1. (20 pts) Let $X = \prod_{\alpha \in J} X_\alpha$ be a non–empty product of topological spaces and $b \in X$. Show that
$$A = \{a \in X \mid a_\alpha = b_\alpha \text{ for all but finitely many } \alpha \in J\}$$
is dense in X.

2. (20 pts) **Definition:** A *simple chain* connecting two points a and b of in a topological space X is a collection of open sets $U_1, U_2, \dots, U_n$ such that $a \in U_1$ only, $b \in U_n$ only, and $U_i \cap U_j \neq \emptyset$ if and only if $|i - j| \leq 1$.

   Prove the following theorem.

   Theorem: If X is a connected space and $\mathcal{U}$ is an open cover of X, then any two points a and b can be connected by a simple chain consisting of elements of $\mathcal{U}$.

3. (15 pts) Construct an explicit compactification of $(0, 1)$ for which the functions $f(x) = \sin(1/x)$ and $g(x) = \cos(1/x)$ have continuous extensions.

   Note: Your solution must be explicit; do not simply quote the Stone Cech compactification theorem.

4. (20 pts) Let X be a Hausdorff space with a countable basis. Show that X is compact if it is sequentially compact.

5. (25 pts) Let $\{X_n, d_n\}_{n=1}^\infty$ be a countable family of metric spaces, and suppose that $d_n(x_n, y_n) \leq 1$ for all $n \in \mathbb{Z}_+$ and x_n, $y_n \in X_n$.

   Write $X = \Pi_{n=1}^\infty X_n$, and define a $\rho : X \times X \to \mathbb{R}$ by
$$\rho(x, y) = \sup \left\{ \frac{d_n(x_n, y_n)}{n} \mid n \in \mathbb{Z}_+ \right\}$$
   where we write $x = (x_n)_{n=0}^\infty$ and similarly for y.

   (a) Show that ρ is a metric.

   (b) Prove that the metric topology $\mathcal{T}_\rho$ is coarser than the product topology $\mathcal{T}$ on X.

   (c) Prove that the metric topology $\mathcal{T}_\rho$ is finer than the product topology $\mathcal{T}$ on X.