BASIC EXAMINATION: GENERAL TOPOLOGY

Thursday September 9, 2021, 5:30pm-8:30pm

Time allowed: 180 minutes.
This test is closed book: no notes or other aids are permitted. You may use without proof standard results from the syllabus which are independent of the question asked, unless explicitly instructed otherwise. You must, however, clearly state the result you are using.

1. Let (X, τ) be a topological space.

(i) [**4 points.**] Give the definition of:

1. (X, τ) is Hausdorff;
2. (X, τ) is regular;
3. (X, τ) is completely regular;
4. (X, τ) is normal.

(ii) [**14 points.**] Prove that if (X, τ) is compact and Hausdordff then it is normal.

2. [**14 points.**] Show that a set $K \subset \ell^2(\mathbb{N})$ is compact if and only if it is closed, bounded, and for every $\varepsilon > 0$ there exists $m = m(\varepsilon)$ such that for all $\{x_n\}_{n\in\mathbb{N}} \in K$

$$\sum_{n=m}^{\infty} x_n^2 < \varepsilon.$$

3.

(i) [**4 points**] Let (X, τ) be a topological space and let $E \subset X$. When do we say that E is connected?

(ii) [**14 points**] Let (X, d) be a metric space and let E_1 and E_2 be connected sets such that $E_1 \cap \overline{E_2} \neq \emptyset$. Prove that $E_1 \cup E_2$ is connected.

(iii) [**14 points**] Prove that if (X, τ) is a topological space and $E \subset X$ is connected then $\overline{E}$ is connected.

4.

(i) [**4 points**] State Urysohn's Lemma.

(ii) [**14 points.**] Prove that if C is a nonempty closed set strictly contained in X, then there exists a continuous function $f : X \to [0, 1]$ such that $f \equiv 0$ on C and $f > 0$ in $X \setminus C$ if and only if C is a G_δ set. **(OVER)**

5.

(i) [**4 points**] State Tietze's Extension Theorem.

(ii) [**14 points**] Let (X, τ) be a Hausdorff completely regular topological space. Prove that if K is a compact subset and $f : K \to \mathbb{R}$ is continuous then there exists a continuous function $g : X \to \mathbb{R}$ such that $g\big|_K = f\big|_K$.