TOPOLOGY BASIC EXAM

Monday February 1st, 6:30pm-9:30pm EST

Zoom ID: 962 7897 1453
Passcode: 260002

You are requested to keep your video on during the entire time until you submit your solutions. The exam will be recorded.

Closed books, notes. Access to any other relevant materials in any way, including INTERNET consultation, is not allowed.

Please upload your solutions here:
https://cmu.app.box.com/f/9876e0530f0c48b98ef1acef53d72962

1. [**20 points.**] Let ℓ^∞ be the space of bounded sequences of real numbers, equipped with the $||\cdot||_\infty$ norm, i.e., if $x \in \ell^\infty$ is the sequence $x = (x_1, x_2, \ldots)$, then

$$||x||_\infty := \sup_{n\in\mathbb{N}} |x_n|.$$

Let c be the set of all convergent sequences, and let c_0 be the set of sequences in c with limit zero. Prove that c and c_0 are closed subspaces of ℓ^∞ .

2. [**20 points.**] Consider $\mathbb{R}^{\mathbb{N}} = \Pi_{n\in\mathbb{N}} X_n$, where $X_n = \mathbb{R}$ for all $n \in \mathbb{N}$, i.e., the space of all sequences $x = (x_1, x_2, \ldots)$, $x_n \in \mathbb{R}$ for all $n \in \mathbb{N}$. Let Y be the subset of $\mathbb{R}^{\mathbb{N}}$ consisting of all sequences that are "eventually zero", that is, all sequences $x = (x_1, x_2, \ldots)$ such that $x_n \neq 0$ for only finitely many $n \in \mathbb{N}$. What is the closure of Y in the box and product topologies?

3. [**20 points.**] Let (X, τ) be a completely regular topological space, and consider its Stone–Čech compactification $\beta(X)$. Prove that X is connected if and only if $\beta(X)$ is connected.

4. (i) [**5 points.**] State Baire's Category Theorem.
(ii) [**15 points.**] Let X be a complete metric space, and let $\mathcal{F}$ be a subset of $C(X;\mathbb{R})$ such that for every $x \in X$, the set $\{f(x) : f \in \mathcal{F}\}$ is bounded. Prove that there is a nonempty open set U of X in which the functions in $\mathcal{F}$ are uniformly bounded, i.e., there exist $M \in \mathbb{R}$ such that $\sup_{f\in\mathcal{F},x\in U} |f(x)| = M$.

5. (i) [**5 points.**] State Ascoli-Arzelà Theorem.
(ii) [**15 points.**] Let X be a compact metric space, and let $f_n \in C(X;\mathbb{R})$ be an equicontinuous and pointwise bounded sequence of functions. Prove that if every uniformly convergent subsequence has the same limit $f \in C(X;\mathbb{R})$, then f_n converge uniformly to f.