DEPARTMENT OF MATHEMATICAL SCIENCES
CARNEGIE MELLON UNIVERSITY

BASIC EXAMINATION
GENERAL TOPOLOGY
SEPTEMBER 2020

Time allowed: 3 hours.

1. (15 points) Let X be an infinite set. Let
$$\tau = \{U \subseteq X \ : \ X \backslash U \text{ is finite}\} \cup \{\emptyset\}.$$
It is known that τ is a topology.
   (i) Show that if X is uncountable then τ is not first countable (i.e. does not satisfy the first axiom of countability).
   (ii) Let $x \in X$. Show that a sequence $\{x_n\}_{n=1,2,\ldots}$ converges to x if and only if for each $n \in \mathbb{N}$ either $x_n = x$ or there exists $n_0 \in \mathbb{N}$ such that for all $m > n_0$ $x_m \neq x_n$. (In other words no element except x can appear infinitely many times in the sequence.)

2. (25 points) We define a topology τ on $\mathbb{R}^2$ as follows: subset $U \subset \mathbb{R}^2$ beelongs to τ if at every point $x \in U$, U contains an open line segment through x in every direction, that is for every $v \in S^1$ there exists $\varepsilon > 0$ such that for every $s \in (-\varepsilon, \varepsilon), \quad x + sv \in U$.
   (i) Prove that τ is a topology. What is the relation to the standard topology (weaker, stronger, neither)? What is the induced topology on any straight line of $\mathbb{R}^2$? What is the induced topology on a circle?
   (ii) Prove that $(\mathbb{R}^2, \tau)$ is separable and Hausdorff.
   (iii) Prove that there exists closed set $E \subset \mathbb{R}^2$ which is equipotent to $\mathbb{R}$ and is such that the induced topology is discrete. Prove that $(\mathbb{R}^2, \tau)$ is not normal.

3. (15 points) Let $f : X \to Y$ be a continuous and closed mapping. Assume that Y is compact and that for all $y \in Y$, $f^{-1}(\{y\})$ is compact. Show that X is compact.

4. (25 points) Let $X = C([a,b],\mathbb{R})$ for some $a < b$.

   (i) Show that for $p > 0$, $d_p : X \times X \to \mathbb{R}$ defined by

   $$d_p(f,g) = \max_{t\in[a,b]} |f(t) - g(t)|e^{-pt}$$

   is a metric on X.

   (ii) Show that for any $p > 0$ the metric d_p generates the same topology on X as the standard metric on X:

   $$d(f,g) = \max_{t\in[a,b]} |f(t) - g(t)|.$$

   (iii) Let $h \in X$ and $K \in C([a,b]\times[a,b],\mathbb{R})$. Show that there exists a unique $f \in X$ which satisfies the equation

   $$f(t) = h(t) + \int_a^t K(t,s)f(s)ds \quad \text{for all } t \in [a,b].$$

   Hint: Use Banach contraction principle in (X, d_p) for appropriately chosen $p > 0$.

5. (20 points) Consider the metric space $X = C\left([0,1],\mathbb{R}\right)$ with the sup metric:

   $$d_\infty(f,g) = \max_{x\in[0,1]} |f(x) - g(x)|.$$

   For every $n \in \mathbb{N}$ let

   $$X_n := \{f \in X : \text{ there is } x \in [0,1] \text{ such that } \\ |f(x) - f(y)| \leq n|x-y| \text{ for all } y \in [0,1]\}.$$

   (i) Fix $n \in \mathbb{N}$ and prove that each $f \in X$ can be approximated by a zigzag (piecewise linear) function $g \in X$ with sufficiently large slopes so that it does not belong to X_n and such that $d_\infty(f,g)$ is arbitrarily small.

   (ii) Fix $n \in \mathbb{N}$ and prove that every open set $U \subset X$ contains an open set that does not intersect X_n.

   (iii) Prove that there exists a dense G_δ set in X that consists of nowhere differentiable functions. A set is a G_δ set if it is a countable intersection of open sets.