DEPARTMENT OF MATHEMATICAL SCIENCES
CARNEGIE MELLON UNIVERSITY

BASIC EXAMINATION
GENERAL TOPOLOGY
JANUARY 2020

Time allowed: 3 hours.

Name: ____

Problem	Points
1 (20)	
2 (20)	
3 (20)	
4 (20)	
5 (20)	
Total (100)	

1. Show that any compact metric space has a countable base of its topology.

2. For all $s \in \mathbb{R}$, let (X_s, τ_s) be a topological space where the topology is not trivial. Show that the product space

$$X = \Pi_{s \in \mathbb{R}} X_s$$

endowed with product topology is not first countable.

3. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of functions in $C^1([0,1],\mathbb{R})$ such that for all n

$$\int_0^1 f_n(x)dx = 0$$

and $|f_n'(x)| \leq \frac{1}{\sqrt{x}}$ for all $x \in (0,1]$. Prove that $\{f_n\}_{n\in\mathbb{N}}$ has a uniformly convergent subsequence.

4. Consider $\mathbb{R}^{\mathbb{N}} = \{\{x_k\}_{k\in\mathbb{N}} \ : \ x_k \in \mathbb{R}\}$. Let $\{\{x_k^n\}_{k\in\mathbb{N}}\}_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}^{\mathbb{N}}$. Show that $x^n \to p$ in the box topology as $n \to \infty$ iff

   (i) for all $k \in \mathbb{N}$ $x_k^n \to p_k$ as $n \to \infty$ and

   (ii) there exist $I \subset \mathbb{N}$ finite and $n_0 \in \mathbb{N}$, such that for all $k \in \mathbb{N} \setminus I$ and all $n \geq n_0$ it holds that $x_k^n = p_k$.

   Show that $\mathbb{R}^{\mathbb{N}}$ with the box topology is not pathwise connected. Furthermore show that x, y belong to the same component of pathwise connectedness iff $x_n = y_n$ for all but finitely many indices.

5. Consider the set of natural numbers $\mathbb{N}$ with the discrete topology. Let $\beta(\mathbb{N})$ be the Stone-Čech compactification of $\mathbb{N}$. Show that $\beta(\mathbb{N})$ is not sequentially compact.