BASIC EXAMINATION: GENERAL TOPOLOGY

Tuesday August 27, 2019, 4:30pm-7:30pm

1.

(i) **[8 points]** Let (X, τ) be a topological space. Give the definition of:

- (a) (X, τ) is separable;
- (b) (X, τ) is normal;
- (c) (X, τ) is connected;
- (d) (X, τ) is compact.

(ii) **[8 points]** Prove that ℓ^∞ is not separable.

2.

(i) **[5 points]** Given a collection of topological spaces $\{(X_\alpha, \tau_\alpha)\}_{\alpha \in \Lambda}$, give the definition of product topology and box topology.

(ii) **[5 points]** State Tikhonov's Theorem.

(iii) **[10 points]** Prove that $\prod_{\alpha \in \Lambda} X_\alpha$ is a Hausdorff space if and only if each $(X_\alpha, \tau_\alpha)_{\alpha \in \Lambda}$ is a Hausdorff topological space.

(iv) **[8 points]** Prove that $[0,1]^{\mathbb{N}}$ with the box topology is not compact.

3. [15 points] Let (X, τ_X) and (Y, τ_Y) be topological spaces, let $A \subset X$ and $B \subset Y$ be compact sets, and let $W \in \tau_{X \times Y}$ be an open set such that $A \times B \subset W$. Prove that there exist open sets $U \in \tau_X$, $V \in \tau_Y$, such that

$$A \times B \subset U \times V \subset W.$$

4.

(i) **[5 points]** Give the definition of Baire space.

(ii) **[13 points]** State and prove Baire Category Theorem.

(iii) **[8 points]** Prove that the vector space of all polynomials in one variable with real coefficients is not a Banach space in any norm.

5.

(i) **[5 points]** State Ascoli-Arzelà Theorem.

(ii) **[10 points]** Let $\{f_n\} \subset C^2((0,1))$ be a sequence of functions such that

$$\sup_{n \in \mathbb{N}} \{|f_n(0)| + |f_n'(0)|\} =: M < \infty,$$

and there exists $\alpha \in (1,2)$ such that for all $n \in \mathbb{N}$ and $x \in (0,1)$

$$|f''(x)| \leq \frac{1}{(1-x)^\alpha}.$$

Prove that $\{f_n\}$ admits a subsequence that converges to a continuous function uniformly on compact sets.