BASIC EXAMINATION: GENERAL TOPOLOGY

Thursday January 24, 2019, 4:30pm-7:30pm

1. **[12 points]** Let (X, τ) be a topological space. Give the definition of:

(i) (X, τ) is separable;

(ii) (X, τ) is normal;

(iii) (X, τ) is complete;

(iv) (X, τ) is compact;

(v) (X, τ) is paracompact;

(vi) (X, τ) is a Baire space.

2. Let (X, τ) be Hausdorff, locally compact and second countable.

(i) **[11 points]** Prove that there exist open sets U_n with $\overline{U_n}$ compact such that

$$\overline{U_n} \subset U_{n+1} \quad \text{and } X = \cup_{n=1}^{\infty} U_n.$$

(ii) **[13 points]** Prove that (X, τ) is paracompact.

3.

(i) Let (X, d) be a metric space that satisfies the following for every continuous function $f : X \to \mathbb{R}$:

Property: For every every $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $A \subset X$ has diam(A)$< \delta$ then there exists a finite set $B \subset \mathbb{R}$ such that for all $a \in A$ there exists $b \in B$ with $f(a) \in B(b, \varepsilon)$.

Let $\mathcal{U}$ be an open cover of X. $\mathcal{U}$ is said to have a *weak Lebesgue number* $\delta > 0$ if for every $A \subset X$ with diam(A)$< \delta$ there exist $U_i \in \mathcal{U}$, for $i \in \{1, \ldots, n\}$ and some $n \in \mathbb{N}$, with $A \subset \cup_{i=1}^{n} U_i$.

(a) **[8 points]** Let $f : X \to \mathbb{R}$ be a continuous function, and let $\varepsilon > 0$. Prove that the open covering $\{f^{-1}((q - \varepsilon, q + \varepsilon)) : q \in \mathbb{Q}\}$ has a weak Lebesgue number.

(b) **[8 points]** Prove that if Y is a closed subspace of X, then Y satisfies the Property above.

(c) **[12 points]** Prove that X is complete.

(ii) **[12 points]** Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \to Y$ satisfy the Property above with Y in place of $\mathbb{R}$. Let $\{x_n\}_{n \in \mathbb{N}}$ be a Cauchy sequence in X. Prove that $\{f(x_n)\}_{n \in \mathbb{N}}$ admits a Cauchy subsequence.

4. **[12 points]** State and prove Baire Category Theorem.

5. **[12 points]** Let (X, d) be a compact metric space. Prove that $C(X; \mathbb{R})$ is separable.