Department of Mathematical Sciences
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BASIC EXAMINATION: GENERAL TOPOLOGY

August 30, 2018, 4:30pm-7:30pm, WEH 7201

1. Let (X, τ) be a topological space.

(i) [**10 points.**] Give the definition of:

1. (X, τ) is T_1;
2. (X, τ) is Hausdorff;
3. (X, τ) is regular;
4. (X, τ) is completely regular;
5. (X, τ) is normal.

(ii) [**5 points.**] Prove that if X is compact, T_1, and satisfies the first axiom of countability, then X is sequentially compact.

(iii) [**6 points.**] Prove that if X is compact and Hausdorff then it is regular.

(iv) [**6 points.**] Prove that if X is compact and Hausdorff then it is normal.

2. Let (X, d) be a sequentially compact metric space.
(a) [**9 points**] Prove that (X, d) is separable. Conclude that every sequentially compact metric space is second countable.
(b) [**8 points**] Prove that (X, d) is *Lindelöf*, i.e., every open cover admits a countable subcover.
(c) [**8 points**] Using (b) prove that (X, d) is compact.

3. [**17 points.**] Let (X, d_X) and (Y, d_Y) be metric spaces and let

$$C_b(X;Y) := \{f \in C(X;Y) : \sup_{x,z \in X} d_Y(f(x), f(z)) < +\infty\}$$

equipped with the metric $d(f, g) := \sup_{x \in X} d_Y(f(x), g(x)$, for $f, g \in C_b(X;Y)$. Prove that $(C_b(X : Y), d)$ is complete if and only if (Y, d_Y) is complete.

4.

(a) Give the definition of

(i) [**4 points**] connected topological space;

(ii) [**4 points**] path connected topological space.

(b) [**9 points**] Prove that if (X, τ) is a topological space and $E \subset X$ is connected then $\overline{E}$ is connected.

5. Let $E \subset [0, 1]$ and let

$$G := \{f \in C([0, 1]) : f(x) = 0 \text{ for all } x \in E\}.$$

(a) [**7 points**] In $C([0, 1])$ consider the metric

$$d_\infty : (f, g) = \max_{x \in [0,1]} |f(x) - g(x)|.$$

Prove that the set G is closed in $(C([0, 1]), d_\infty)$.

(b) [**7 points**] Find a necessary and sufficient condition on E for G to be bounded in $(C([0, 1]), d_\infty)$.