DEPARTMENT OF MATHEMATICAL SCIENCES
CARNEGIE MELLON UNIVERSITY

BASIC EXAMINATION: GENERAL TOPOLOGY

January 22, 2018, 4:30pm-7:30pm

1. (a) [**6 points**] Give the definition of a connected topological space.

(b) [**10 points**] Let (X, τ_X) and (Y, τ_Y) be topological spaces. Prove that $X \times Y$ is connected if and only if X and Y are connected.

2. (a) [**5 points**] Give the definition of a completely regular topological space.

(b) [**6 points**] Give the definition of a compact metric space.

(c) [**6 points**] State Urysohn's Lemma.

(d) [**6 points**] State Tietze's Extention Theorem

(e) [**11 points**] Prove that (X, τ) is completely regular if and only if for nonempty disjoint subsets K and C with K compact and C closed there exists a continuous function $f : X \to [0, 1]$ such that $f\big|_K \equiv 0$ and $f\big|_C \equiv 1$.

3. (a) [**6 points**] Give the definition of a separable topological space.

(b) [**5 points**] Give the definition of a Lindelöf space.

(c) [**11 points**] Prove that a metric space is a Lindelöf space if and only if it satisfies the second axiom of countability.

4. [**11 points**] Prove that a complete metric space (X, d) is compact if and only if for every $\varepsilon > 0$ there exists a finite set $K \subset X$ such that $d(x, K) < \varepsilon$ for every $x \in X$.

5. Let $E := \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0\}$ and let $f_n : E \to \mathbb{R}$ be defined by

$$f_n(x, y) := \frac{n^2}{x^n + y^n + ny}, \quad (x, y) \in E.$$

(a) [**6 points**] Find the largest set $E_0 \subset E$ where there is pointwise convergence.

(b)[**6 points**] Is there uniform convergence in E_0 ?

(c) [**5 points**] If the answer to (b) is no then find a subset of E with nonempty interior in which there is uniform convergence