DEPARTMENT OF MATHEMATICAL SCIENCES
CARNEGIE MELLON UNIVERSITY

BASIC EXAMINATION
GENERAL TOPOLOGY
AUGUST 2017

Time allowed: 3 hours.

Name: ____

Problem	Points
1 (20)	
2 (20)	
3 (20)	
4 (20)	
5 (20)	
Total (100)	

1. Consider the following topological space. Let $I = [0,1]$ and let $X = I \times \{1,2\}$. Let

$$\mathcal{B} = \{(B(x,r) \times \{1,2\}) \backslash \{(x,2)\} \ : \ x \in I, r > 0\} \cup \{\{(x,2)\} \ : \ x \in I\} \cup \{\emptyset\}.$$

Here $B(x,r)$ is the interval $(x-r, x+r) \cap I$. We note that $\mathcal{B}$ is a basis of topology; call it τ. Show that

(i) (X,τ) is compact.

(ii) (X,τ) is normal.

(iii) (X,τ) is first countable, but not second countable.

2. Let $Y = \Pi_{x \in [0,1]} \mathbb{R}$ (that is $Y = \mathbb{R}^{[0,1]}$). Consider the product topology on Y. Consider the subset E of Y that consists of all functions f that take value zero at a finite number of $x \in [0, 1]$ and that take value 1 otherwise.

   (i) Prove that the function $f_0 \equiv 0$ belongs to $\overline{E}$.

   (ii) Prove that $f_0 \notin \overline{E}^{\text{seq}}$ (the sequential closure of E).

   (iii) Find a function $\Phi : Y \to \mathbb{R}$ such that Φ is sequentially lower-semicontinuous, but not lower semicontinuous. Prove that it has the desired property.

3. Let (X, τ_X) and (Y, τ_Y) be topological spaces and let τ be the product topology on $X \times Y$. Show that if X and Y are compact spaces then the product $(X \times Y, \tau)$ is compact too.

4. Let (X, τ) be a separable normal topological space and let $A \subset X$ be closed. Show that if the induced topology on A is discrete then A is at most countable.

5. Let $f \in C([0,1],\mathbb{R})$ be such that

$$\int_0^1 f(x)x^{2n}dx = 0$$

for all $n \in \mathbb{N} \cup \{0\}$. Show that $f(x) = 0$ for all $x \in [0,1]$.

[Hint: Show that f can be approximated by polynomials of the form $P_n(x^2)$ (so only even powers are used). Then consider $\int_0^1 (f(x) - P_n(x^2))^2 dx$.]