Department of Mathematical Sciences
Carnegie Mellon University

Basic Examination
General Topology
January 2017

Time allowed: 3 hours.

Problem	Points
1 (20)	
2 (20)	
3 (20)	
4 (20)	
5 (20)	
Total (100)	

1. Consider $\mathbb{R}^{\mathbb{N}}$ endowed with the box topology, that is the topology with the base

$$\mathcal{B} = \left\{ \prod_{n=1}^{\infty} U_i \ : \ U_i \text{ is open in } \mathbb{R} \right\}.$$

Show that $f, g \in \mathbb{R}^{\mathbb{N}}$ belong to the same connected component of $\mathbb{R}^{\mathbb{N}}$ if and only if there exists n_0 such that for all $n \geq n_0$, $f_n = g_n$.

2. Let $(X, \prec)$ be a well-ordered uncountable set. Recall that a set is well ordered if it is linearly ordered and every nonempty subset has a minimal element. We define the intervals

$$(a, b) := \{x \in X \ : \ a \prec x \prec b\}$$
$$[a, b] := \{x \in X \ : \ a \preccurlyeq x \preccurlyeq b\}$$

Let 0 be the smallest element of X. Let $\omega_1 = \min\{x \in X : [0, x] \text{ is uncountable}\}$. Consider the *order topology* on $Y = [0, \omega_1)$, that is the topology with the basis

$$\mathcal{B} = \{(a, b) \ : \ a, b \in Y\} \cup \{[0, b) \ : \ b \in Y\}.$$

Show that

(i) Y is not compact.

(ii) Y is countably compact, that is every countable open cover has a finite subcover.

Hint: To show (ii) it is useful to show that Y does not have a closed countable infinite subset on which the induced topology is discrete. To do so assume A is such a subset. Let $\beta = \min\{x \in Y \ : \ [0, x] \cap A \text{ is infinite }\}$. Show that β is a limit point of A.

3. (i) [15 points] Assume that X is a T_1 space such that for every C closed and every W open such that $C \subseteq W$ there exists a sequence of open sets $\{W_n\}_{n=1,2,\dots}$ such that

$$C \subseteq \bigcup_{n=1}^{\infty} W_n \quad \text{and} \quad (\forall n)\ \overline{W}_n \subseteq W.$$

Show that X is T_4.

(ii) [5 points] Show that any 2nd countable T_3 space is T_4.

4. Let $X = C([0,1], \mathbb{R})$. Define for $f, g \in X$

$$d(f,g) = \int_0^1 \frac{|f(x) - g(x)|}{1 + |f(x) - g(x)|}\, dx.$$

(i) Show that d is a metric on X.

(ii) Show that (X, d) is not a complete metric space.

5. Let $\{f_n\}_{n=1,2,\ldots}$ be a sequence of continuous functions on $[0,1]$ with values in $\mathbb{R}$. Assume that f_n are differentiable on $(0,1]$ and assume that for all n and all $x \in (0,1]$

$$|f_n'(x)| \leq \frac{1}{\sqrt{x}}$$

Also assume $\int_0^1 f_n(x)dx = 0$. Prove that $\{f_n\}_{n=1,2,\ldots}$ has a uniformly convergent subsequence.