Department of Mathematical Sciences
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Basic Examination
General Topology
August 2016

Time allowed: 180 minutes.
All problems carry the same weight.

1. (i) Show that the product of any (nonempty) family of nonempty Hausdorff topological spaces is Hausdorff.

   (ii) Show that the product of any (nonempty) family of nonempty completely regular topological spaces is completely regular.

2. (i) Let (X, d) be a metric space. Show that $E \subset X$ is disconnected if and only if there exist U_1 and U_2 **open in X** such that $E \subset U_1 \cup U_2$, $U_1 \cap E \neq \emptyset$, and $U_2 \cap E \neq \emptyset$ and $U_1 \cap U_2 = \emptyset$.

   (ii) Construct a topological space X and a disconnected subset E for which the above is not true.

3. Let $f : X \to Y$ be a continuous and closed mapping. Assume that Y is compact and that for all $y \in Y$, $f^{-1}(\{y\})$ is compact. Show that X is compact.

   Hint: It may be useful to show that for any $y \in Y$ and any open set $U \subseteq X$ containing $f^{-1}(y)$ there exists W_y an open neighborhood of y such that $f^{-1}(W_y) \subseteq U$.

4. Let (X, τ) be a topological space and let $f : X \to \mathbb{R}$. The *epigraph* of f is the set

$$\operatorname{epi} f := \{(x, t) \in X \times \mathbb{R} : f(x) \leq t\}.$$

   Show that f is lower-semicontinuous (LSC) if and only if $\operatorname{epi} f$ is closed.

   Recall that f is LSC if for all $a \in \mathbb{R}$, $f^{-1}((a, \infty))$ is open.

5. Let $\mathcal{M} = \{f : [0,1] \to [0,1] \ : \ f \text{ is nondecreasing}\}$. (Note that functions in $\mathcal{M}$ are not assumed to be continuous.)

   (i) Show that every sequence in $\{f_n\}_n$ in $\mathcal{M}$ has a subsequence that converges pointwise to some function $f \in \mathcal{M}$.

   (ii) Show that if f above is continuous then the convergence of the subsequence is uniform.