Department of Mathematical Sciences
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# Basic Examination
# General Topology
## January 2016

**Time allowed: 120 minutes.**
**Do four of the five problems. Indicate on the first page which problems you have chosen to be graded. All problems carry the same weight.**

1. Let $A$ be a countable subset of $\mathbb{R}^2$. Show that $R^2 \backslash A$ is pathwise connected.

2. Let $f : X \to Y$ be a continuous, closed and surjective. Assume that for all $y \in Y$, $f^{-1}(\{y\})$ is compact. Show that

   (i) If $X$ is Hausdorff, so is $Y$.

   (ii) If $X$ is locally compact, so is $Y$.

3. Let $(X, \tau_X)$ and $(Y, \tau_Y)$ be topological spaces and let $f : X \to Y$ be a mapping. Let $\Gamma = \{(x, y) \in X \times Y \,:\, y = f(x)\}$ be the graph of $f$.

   (i) Show that if $Y$ is Hausdorff and $f$ is continuous then $\Gamma$ is closed in $(X \times Y, \tau_X \times \tau_Y)$.

   (ii) Assume $X = Y$. Show that if $X$ is not Hausdorff then for the identity function, $f(x) = x$ for all $x \in X$, the graph is not closed.

4. Let $(X, \tau)$ be a $T_{3\frac{1}{2}}$ space. Show that $X$ is homeomorhpic to a subset of $Y = [0, 1]^I$ for some nonempty set $I$. The topology on $Y$ is the product topology where on $[0, 1]$ we consider the standard topology.

5. Let $X = \{f \in C([0, 1], \mathbb{R}) \,:\, f(0) = f(1) = 0\}$. Let $\mathcal{B} \subset X$ be the set of functions on $[0, 1]$ of the form $x \mapsto x^n(1 - x)$ where $n \geq 1$. Let $\mathcal{A} = \operatorname{span} \mathcal{B}$ where by span we mean the set of all linear combinations of functions in $\mathcal{B}$. Show that $\mathcal{A}$ is dense in $X$.