Department of Mathematical Sciences
Carnegie Mellon University

Basic Examination
General Topology
August 2015

Time allowed: 120 minutes.
Do four of the five problems. Indicate on the first page which problems you have chosen to be graded. All problems carry the same weight.

1. Let $X = \mathbb{R}^{\mathbb{N}}$ be the set of sequences of real numbers. Consider the uniform topology on X, that is the topology given by the metric
$$d(\mathbf{x}, \mathbf{y}) = \inf\{1, \sup_{n\in\mathbb{N}} |x_n - y_n|\}.$$
Show that $\mathbf{x}$ and $\mathbf{y}$ belong to the same connected component of X if and only if the sequence $\mathbf{x} - \mathbf{y}$ is bounded (in the standard metric of $\mathbb{R}$).

2. Let $\{(X_\alpha, \tau_\alpha)\}_{\alpha\in\Lambda}$ be a collection of nonempty topological spaces, and let $E_\alpha \subset X_\alpha$ be nonempty for every $\alpha \in \Lambda$. Fix $g \in \prod_{\alpha\in\Lambda} E_\alpha$ and consider the set
$$E := \left\{ f \in \prod_{\alpha\in\Lambda} E_\alpha : \ f(\alpha) = g(\alpha) \text{ for all but finitely many } \alpha \in \Lambda \right\}.$$
Prove that
$$\overline{E} = \overline{\prod_{\alpha\in\Lambda} E_\alpha}.$$

3. Let (X, τ) be a metrizable Lindelöf topological space, that is space such that every open cover of X has a countable subcover. Show that X is separable.

4. A space is said to be locally compact if every point has a compact neighborhood. Show that every open subset of a locally compact Hausdorff space is locally compact.

5. Let $f \in C([0,1], \mathbb{R})$ be such that $f(0) = 0$. Show that for any $\varepsilon > 0$ there exists an odd polynomial, that is one with only odd powered monomials:
$$P(x) = \sum_{i=1}^{n} a_{2i-1} x^{2i-1}$$
such that $d(f, P) = \max_{x\in[0,1]} |f(x) - P(x)| < \varepsilon$.