Department of Mathematical Sciences
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Basic Examination
General Topology
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Time allowed: 120 minutes.
Do four of the five problems. Indicate on the first page which problems you have chosen to be graded. All problems carry the same weight.

1. Prove that $[0,1]^{\mathbb{N}}$ with the box topology is not compact. [Box topology on a product is the smallest topology in which any product of open sets is open.]

2. Let (X,τ) be a topological space and let $f : X \to \mathbb{R}$. Show that f is lower semicontinuous if and only if $\{x \in X : f(x) \leq t\}$ is closed for every $t \in \mathbb{R}$.

   Recall that $f : X \to \mathbb{R}$ is *lower semicontinuous* at a point $x_0 \in X$ if either x_0 is an isolated point or x_0 is an accumulation point (that is limit point) of X and

   $$\liminf_{x \to x_0} f(x) \geq f(x_0).$$

   The function f is said to be *lower semicontinuous* if it is lower semicontinuous at every point of X.

3. Let (X,τ) be a compact Hausdorff space. Let $p \in X$. Assume there exists a countable family of open sets $\{U_i : i \in \mathbb{N}\}$ such that $\cap_{i\in\mathbb{N}} U_i = \{p\}$. Prove that there exists a countable local base at p.

4. Consider $A = \{-1/n : n = 1, 2, \dots\} \cup \{0\}$. Let $X = C(A,\mathbb{R})$ with $d_X(f,g) = \sup_{x\in A} |f(x) - g(x)|$. Let $Y = l^\infty(\mathbb{N})$. Show that the spaces X and Y are not homeomorphic.

5. Let $X = C([a,b],\mathbb{R})$ for some $a < b$.

   (a) Show that for $p > 0$, $d_p : X \times X \to \mathbb{R}$ defined by

   $$d_p(f,g) = \max_{t\in[a,b]} |f(t) - g(t)| e^{-pt}$$

   is a metric on X.

(b) Show that for any $p > 0$ the metric d_p generates the same topology on X as the standard metric on X:

$$d(f, g) = \max_{t \in [a,b]} |f(t) - g(t)|.$$

(c) Let $h \in X$ and $K \in C([a, b] \times [a, b], \mathbb{R})$. Show that there exists unique $f \in X$ which satisfies the equation

$$f(t) = h(t) + \int_a^t K(t, s) f(s) ds \quad \text{for all } t \in [a, b].$$

Hint: Use Banach contraction principle in (X, d_p) for appropriately chosen $p > 0$.

6. Let $\{f_n\}_{n=1,2,\ldots}$ be a uniformly bounded, equicontinuous sequence of real-valued functions on compact metric space (X,d). Define $g_n : X \to \mathbb{R}$ by

$$g_n(x) = \max\{f_1(x), \ldots, f_n(x)\}.$$

Prove that the sequence $\{g_n\}_{n=1,2,\ldots}$ converges uniformly.

7. Let (X, τ_X) and (Y, τ_Y) be topological spaces and let $f : X \to Y$ be a mapping. Let $\Gamma = \{(x,y) \in X \times Y \ : \ y = f(x)\}$ be the graph of f

   (i) Show that if Y is Hausdorff and f is continuous then Γ is closed in $(X \times Y, \tau_X \times \tau_Y)$.

   (ii) Assume $X = Y$. Show that if X is not Hausdorff then for the identity function, $f(x) = x$ for all $x \in X$, the graph is not closed.

8. Let $X = \{(x,y) \ : \ x, y \in \mathbb{R}, y \geq 0\}$. Let

$$\mathcal{B} = \{B_r(x,y) : 0 < r \leq y\} \cup \Big\{ \{(a,0)\} \cup (B_r(a,0) \cap \{(x,y) : y > 0\}) : a \in \mathbb{R}, r > 0 \Big\} \cup \{\emptyset\}.$$

It is known that $\mathcal{B}$ is a basis of a topology on X. Show that the topology generated by $\mathcal{B}$ is Hausdorff but not regular. Also show that the topological space is separable, but does not have a countable basis.

9. Consider the space $X = [0,1]^{[0,1]}$ equipped with product topology. Let $K = \{f \in X \ : \ \exists Y -$ countable $(\forall x \in [0,1] \setminus Y)\, f(x) = 0\}$. Show that K is not compact. Show that K is sequentially compact.