DEPARTMENT OF MATHEMATICAL SCIENCES
CARNEGIE MELLON UNIVERSITY

BASIC EXAMINATION
GENERAL TOPOLOGY
AUGUST 2013

Time allowed: 120 minutes.
Do four of the five problems. Indicate on the first page which problems you have chosen to be graded. All problems carry the same weight.

1. Consider
$$X = C((0,\infty)) := \{f : (0,\infty) \to \mathbb{R} : f \text{ is continuous}\}.$$
Let $K_n := \left[\frac{1}{n}, n\right]$. Then
$$\bigcup_{n=1}^{\infty} K_n = (0,\infty).$$
For $f, g \in X$ define
$$d(f,g) := \sup_n \frac{1}{2^n} \frac{\max_{x \in K_n} |f(x) - g(x)|}{1 + \max_{x \in K_n} |f(x) - g(x)|}. \tag{1}$$
Show that (X, d) is a metric space. Then prove that $d(f_n, f) \to 0$ if and only if $f_n \to f$ uniformly on compact sets.

2. Let $\{f_n\}_{n=1,2,\ldots}$ be a uniformly bounded, equicontinuous sequence of real-valued functions on compact metric space (X, d). Define $g_n : X \to \mathbb{R}$ by
$$g_n(x) = \max\{f_1(x), \ldots, f_n(x)\}.$$
Prove that the sequence $\{g_n\}_{n=1,2,\ldots}$ converges uniformly.

3. Let (X, τ_X) and (Y, τ_Y) be topological spaces and let $f : X \to Y$ be a mapping. Let $\Gamma = \{(x, y) \in X \times Y \ : \ y = f(x)\}$ be the graph of f
   (i) Show that if Y is Hausdorff and f is continuous then Γ is closed in $(X \times Y, \tau_X \times \tau_Y)$.
   (ii) Assume $X = Y$. Show that if X is not Hausdorff then for the identity function, $f(x) = x$ for all $x \in X$, the graph is not closed.

4. Let $X = \{(x, y) \,:\, x, y \in \mathbb{R}, y \geq 0\}$. Let

$$\mathcal{B} = \{B_r(x,y) : 0 < r \leq y\} \cup \Big\{ \{(a,0)\} \cup (B_r(a,0) \cap \{(x,y) : y > 0\}) : a \in \mathbb{R}, r > 0 \Big\} \cup \{\emptyset\}.$$

It is known that $\mathcal{B}$ is a basis of a topology on X. Show that the topology generated by $\mathcal{B}$ is Hausdorff but not regular. Also show that the topological space is separable, but does not have a countable basis.

5. Consider the space $X = [0,1]^{[0,1]}$ equipped with product topology. Let $K = \{f \in X \,:\, \exists Y - \text{countable}\ (\forall x \in [0,1] \setminus Y)\ f(x) = 0\}$. Show that K is not compact. Show that K is sequentially compact.