Department of Mathematical Sciences
Carnegie Mellon University

Functional Analysis

Sample Exam

Do any 4 of the following 6 problems. All problems carry equal weight.

1. (a) Let X be a Banach space, Y be a normed linear space, and $\{T_n\}_{n=1}^\infty$ be a sequence of bounded linear mappings from X to Y satisfying

$$\forall\, x \in X, \quad \sup\{\|T_n x\| : n \in \mathbb{N}\} < \infty.$$

Prove that

$$\sup\{\|T_n\| : n \in \mathbb{N}\} < \infty.$$

(Do not simply quote the Banach-Steinhaus Theorem (aka the Principle of Uniform Boundedness). You are being asked to prove that theorem.)

(b) Give an example of a normed linear space X, a Banach space Y, and a sequence $\{T_n\}_{n=1}^\infty$ of bounded linear mappings from X to Y satisfying

$$\forall\, x \in X, \quad \sup\{\|T_n x\| : n \in \mathbb{N}\} < \infty,$$

but

$$\sup\{\|T_n\| : n \in \mathbb{N}\} = \infty.$$

2. (a) Let X be a reflexive Banach space and let $\{K_n\}_{n=1}^\infty$ be a sequence of bounded subsets of X satisfying (i) and (ii) below.

(i) $\forall n \in \mathbb{N},\ K_n \neq \emptyset,\ K_n$ is closed, K_n is convex,

(ii) $\forall n \in \mathbb{N},\ K_{n+1} \subset K_n$.

Show that

$$\bigcap_{n=1}^{\infty} K_n \neq \emptyset.$$

(b) Give an example of a (nonreflexive) Banach space X and a sequence $\{K_n\}_{n=1}^\infty$ of bounded subsets of X satisfying (i) and (ii) above, but with

$$\bigcap_{n=1}^{\infty} K_n = \emptyset.$$

3. Let X be a complex Hilbert space with inner product $(\cdot,\cdot)$ and A be a bounded linear mapping from X to X. Prove that A is compact if and only if $(Ax_n, x_n) \to 0$ as $n \to \infty$ for every sequence $\{x_n\}_{n=1}^\infty$ such that $x_n \rightharpoonup 0$ (weakly) as $n \to \infty$. What happens with regard to this result in real Hilbert spaces?

4. (a) State the Open Mapping Theorem and the Closed Graph Theorem.

   (b) Use the Open Mapping Theorem to Prove the Closed Graph Theorem.

   (c) Let X, Y, Z be Banach spaces and $U : X \to Y$, $V : Y \to Z$ be linear mappings and define $T : X \to Z$ by $Tx = VUx$ for all $x \in X$. Assume that T is continuous and that V is continuous and injective. Prove that U is continuous.

5. (a) Let X be a Banach space and $T : X \to X$ be a linear mapping such that $T^2 = T$. Show that T is continuous if and only if the null space and range of T both are closed.

   (b) Let X, Y be Banach spaces and put

$$\mathcal{O} = \{T \in \mathcal{L}(X; Y) : T^*[Y^*] = X^*\}.$$

   Show that $\mathcal{O}$ is an open subset of $\mathcal{L}(X; Y)$ (equipped with the operator norm). Here $\mathcal{L}(X; Y)$ is the set of all bounded linear mappings from X to Y, X^* and Y^* are the (topological) duals of X and Y, and $T^* \in \mathcal{L}(Y^*; X^*)$ is the adjoint of an operator $T \in \mathcal{L}(X; Y)$.

6. Let X be an infinite-dimensional (real) Banach space. Show that there exist convex sets $K_1, K_2 \subset X$ such that $K_1 \cap K_2 = \emptyset$, $K_1 \cup K_2 = X$, $\mathrm{cl}(K_1) = \mathrm{cl}(K_2) = X$. Here $\mathrm{cl}(S)$ denotes the closure of a set $S \subset X$.