27 Aug 2024 **CMU**

Basic examination: Functional Analysis

<u>*180 min*</u>.

15pts **1.** a) Let $\|\cdot\|$ be a norm on $\mathbb{R}^d$. Show that its unit ball $B = \{x \in \mathbb{R}^d,\ \|x\| \leq 1\}$ is a symmetric (about the origin), compact, convex set with nonempty interior.

b) Conversely, let K be a symmetric (about the origin), compact, convex set with nonempty interior in $\mathbb{R}^d$. Show that there is a norm on $\mathbb{R}^d$ whose closed unit ball is precisely K.

10pts **2.** Show that there is a positive constant C such that for every real polynomial p of degree at most 2024, we have $|p(15)| \leq C \sup_{t\in[0,1]} |p(t)|$.

10pts **3.** Let Y be a proper closed subspace of a real normed space X. Show that there is a bounded linear functional ψ on X of norm 1 which vanishes on Y.

Feel free to use "well-known" results, as long as you state them properly.

20pts **4.** Suppose that $(\phi_n)_{n=1}^{\infty}$ is a sequence of linear functionals on a real Banach space. Assume that $\|\phi_n\| = 4^n$ for each n and choose a vector x_n of norm $\frac{1}{3^n}$ with $|\phi_n(x_n)| \geq \frac{3}{4}\left(\frac{4}{3}\right)^n$.

a) Argue (inductively) that we can choose signs $\varepsilon_1, \varepsilon_2, \dots \in \{-1, 1\}$ so that $\phi_n(\varepsilon_n x_n)$ has the same sign as $\phi_n(\sum_{k=1}^{n-1} \varepsilon_k x_k)$.

b) Let $x = \sum_{k=1}^{\infty} \varepsilon_k x_k$ and show that $|\phi_n(x)| \to \infty$ as $n \to \infty$.

c) State the uniform boundedness principle for linear functionals. Deduce it from what you have just showed in b).

20pts **5.** a) State and prove Bessel's inequality for orthogonal systems in Hilbert spaces.

b) Let (v_n) be a bounded sequence of vectors in a Hilbert space which are orthogonal. Show that (v_n) converges weakly to 0.

25pts **6.** Let $T\colon L_2[0,1] \to L_2[0,1]$ be the operator defined by

$$(Tf)(x) = \int_0^x f(t)\mathrm{d}t, \qquad f \in L_2[0,1] \text{ (real-valued).}$$

a) Give the definition of compact operators and show that T is compact.

b) Find the (Hilbert) adjoint T^* of T.

c) Argue that T^*T is compact, self-adjoint and $\|T^*T\| = \|T\|^2$.

d) Find the largest eigenvalue of T^*T and hence compute the norm of T.

Feel free to use "well-known" results about compact operators and self-adjoint operators on Hilbert spaces, as long as you state them properly.