Department of Mathematical Sciences
Basic Exam in Functional Analysis
August 2022

Instructions:

- Answer each of the 5 questions.
- You may assume all Banach spaces are over the real numbers.
- If E is a Banach space, the dual is denoted by E', and second dual by E''.
- The ball of radius r in a metric space E centered at $x \in E$ is denoted by $B(x, r)$ or $B_E(x, r)$ if multiple spaces are involved. The unit ball is denoted by $B_E \equiv B_E(0, 1)$.
- $\sigma(E, E')$ denotes the weak topology on E, and $\sigma(E', E)$ the weak star topology on E'.
- $\mathcal{L}(E, F)$ denotes the space of continuous linear operators from E to F.

1. Let E be a Banach space.

   (a) (10 pts) Let $\{f_i\}_{i=1}^k \subset E'$ and $\{\gamma_i\}_{i=1}^k \subset \mathbb{R}$ and suppose that

   $$\left|\sum_{i=1}^k \beta_i \gamma_i\right| \leq \|\sum_{i=1}^k \beta_i f_i\|_{E'}, \qquad \text{for all} \qquad \{\beta_i\}_{i=1}^k \subset \mathbb{R}.$$

   Prove that for all $\epsilon > 0$ there exists $x_\epsilon \in E$ such that $\|x_\epsilon\|_E \leq 1$ and

   $$|f_i(x_\epsilon) - \gamma_i| \leq \epsilon, \qquad 1 \leq i \leq k.$$

   (b) (15 pts) Let E be a Banach space and $J : E \to E''$ the canonical embedding. Prove that $J(B_E)$ is dense in $(B_{E''}, \sigma(E'', E'))$.

2. (15 pts) Let E be a real Banach space and $B \subset E$ be a *barrel* containing the origin; that is,

   - $0 \in B$ and B is closed and convex.
   - (Balanced) If $x \in B$ then $\lambda x \in B$ for all $|\lambda| \leq 1$.
   - (Absorbing) For each $x \in E$ there exists $\lambda \in \mathbb{R}$ such that $\lambda x \in B$.

   Prove that there exists $r > 0$ such that $B(0, r) \subset B$.

3. (20 pts) Let E be a Banach space and $T : E \to E'$ be linear and satisfy

$$(T(x), x) \geq 0 \qquad x \in E.$$

Prove that T is continuous.

4. (20 pts) Let E and F be two Banach spaces and $T \in \mathcal{L}(E, F)$ be surjective and have finite dimensional null space $N(T) \subset E$.

   If $|.|$ is a norm on E satisfying $|x| \leq M\|x\|_E$, prove that there exists a constant $C > 0$ such that

$$\|x\|_E \leq C \left(\|T(x)\|_F + |x| \right), \qquad x \in E.$$

5. Let H be a Hilbert space and $\{u_n\}_{n=1}^{\infty} \subset H$ converge weakly to zero; $u_n \rightharpoonup 0$. Inductively construct a sub–sequence $\{u_{n_k}\}_{k=1}^{\infty}$ such that

$$|(u_{n_j}, u_{n_k})| \leq 1/k, \quad \text{for all} \quad k \geq 2, \text{ and } j = 1, 2, \ldots, k-1.$$

   (a) (10 pts) Let $\sigma_p = (1/p) \sum_{j=1}^{p} u_{n_j}$. Show that σ_p converges strongly to a limit as $p \to \infty$.

   (b) (10 pts) Let $\{u_n\}_{n=1}^{\infty}$ be a bounded sequence in H. Prove that there exists a sub–sequence $\{u_{n_j}\}_{j=1}^{\infty}$ such that $\sigma_p = (1/p) \sum_{j=1}^{p} u_{n_j}$ converges strongly to a limit as $p \to \infty$.