Department of Mathematical Sciences
Carnegie Mellon University

Basic Examination
Introduction To Functional Analysis
January 2022

Time allowed: 180 minutes.

This test is closed book: no notes or other aids are permitted. You may use without proof standard results from the syllabus which are independent of the question asked, unless explicitly instructed otherwise. You must, however, clearly state the result you are using.

1. State and prove the open mapping theorem.
2. Let $(X, \|\cdot\|)$ be a Banach space.
   (a) Prove that if $L \in X'$, then there exists $x_0 \in X$ such that $\|x_0\| = 1$ and $|L(x_0)| \geq \frac{1}{2}\|L\|_{X'}$.
   (b) Prove that if X' is separable, then X is separable.
3. Let $(X, \|\cdot\|)$ be an infinite dimensional Banach space.
   (a) Prove that if $L_1, \ldots, L_n \in X'$, then $\bigcap_{i=1}^n \ker L_i$ contains a line.
   (b) Prove that the weak closure of the unit sphere $\partial B(0,1)$ is the closed ball $\{x \in X : \|x\| \leq 1\}$.
4. Let $1 < p < \infty$ and consider the operator $T : L^p([0,1]) \to L^p([0,1])$, given by

$$T(f)(x) := \int_0^x f(t)\,dt, \quad t \in [0,1].$$

   (a) Prove that T is compact.
   (b) Does T admit any eigenvalues? If so, find them.