Basic Examination: Functional Analysis  

• This test is **closed book**: No books, notes, or access to any other relevant materials (including Internet consultation) are permitted.
• You have 3 hours. The exam has a total of 6 questions and 120 points (20 each).
• You may use without proof *standard* results from the syllabus (not from homework) which are independent of the question asked, unless explicitly instructed otherwise. You must, however, clearly **state** the result you are using.

Below, $\ell^p (1 \leq p < \infty)$ is the Banach space of complex-valued sequences $x = (x_k)_{k \in \mathbb{N}}$ such that the norm

$$
\|x\|_p = \left( \sum_{k=1}^{\infty} |x_k|^p \right)^{1/p}
$$

is finite.

1. Suppose $1 < p < \infty$. Let $S$ be the vector space of sequences $x \in \ell^1 \cap \ell^p$ such that $\sum_{k=1}^{\infty} x_k = 0$. Prove that $S$ is dense in $\ell^p$.

2. Let $H$ be a Hilbert space and let $T : H \to H$ be a bounded linear map. Prove that if both $T$ and $T^*$ are bounded below, then $T$ is bijective with bounded inverse. ($T$ bounded below means that for some $c > 0$, $\|Tx\| \geq c \|x\|$ for all $x \in H$.)

3. Let $(a_k)_{k=1}^{\infty}$ be a decreasing sequence of positive reals and let $R = \lim_{k \to \infty} a_k$. Define $T : \ell^2 \to \ell^2$ by $Tx = (0, a_1 x_1, a_2 x_2, \ldots)$, i.e.,

$$(Tx)_1 = 0, \quad \text{and} \quad (Tx)_{k+1} = a_k x_k, \quad k = 1, 2, \ldots$$

(a) Find all eigenvalues $\lambda \in \mathbb{C}$ of $T$ satisfying $|\lambda| \neq R$.
(b) Find all eigenvalues $\lambda \in \mathbb{C}$ of $T^*$ satisfying $|\lambda| \neq R$.
(c) Show that the spectral radius $r(T) = \max\{ |\lambda| : \lambda \in \sigma(T) \} = R$.

4. Let $X$ be a separable Banach space and let $C \subset X^*$ be a closed convex subset of the dual space. Prove that for any $\ell \in X^*$ there is a point $z \in C$ closest to $\ell$, in the sense that

$$
\|z - \ell\|_{X^*} \leq \|y - \ell\|_{X^*} \quad \text{for all } y \in C.
$$

5. Let $(x_n)$ be a sequence in an arbitrary Banach space $X$ and assume $x_n \rightharpoonup x$ as $n \to \infty$. Prove that there exists a sequence $(y_n)$ with $y_n \in \text{span}\{x_1, \ldots, x_n\}$ such that $\|y_n - x\| \to 0$ as $n \to \infty$.

6. If $T : X \to Y$ is a map between Banach spaces, let us say $T$ **preserves weak convergence** if for every weakly convergence sequence $x_n \rightharpoonup x$ in $X$, the sequence $Tx_n \rightharpoonup Tx$ in $Y$. Prove that for any linear map $T : X \to Y$, $T$ is continuous if and only if $T$ preserves weak convergence.