Basic Examination: Functional Analysis   February 2021

- This test is **closed book**: No books, notes, or access to any other relevant materials (including Internet consultation) are permitted.
- You have 3 hours. The exam has a total of 6 questions and 120 points (20 each).
- You may use without proof *standard* results from the syllabus (not from homework) which are independent of the question asked, unless explicitly instructed otherwise. You must, however, clearly **state** the result you are using.

Below, ℓ^p $(1 \le p < \infty)$ is the Banach space of complex-valued sequences $x = (x_k)_{k\in\mathbb{N}}$ such that the norm

$$\|x\|_p = \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{1/p} \qquad \text{is finite.}$$

1. Suppose $1 < p < \infty$. Let S be the vector space of sequences $x \in \ell^1 \cap \ell^p$ such that $\sum_{k=1}^{\infty} x_k = 0$. Prove that S is dense in ℓ^p.

2. Let H be a Hilbert space and let $T\colon H \to H$ be a bounded linear map. Prove that if both T and T^* are bounded below, then T is bijective with bounded inverse. (T bounded below means that for some $c > 0$, $\|Tx\| \ge c\|x\|$ for all $x \in H$.)

3. Let $(a_k)_{k=1}^{\infty}$ be a decreasing sequence of positive reals and let $R = \lim_{k\to\infty} a_k$. Define $T : \ell^2 \to \ell^2$ by $Tx = (0, a_1x_1, a_2x_2, \ldots)$, i.e.,

$$(Tx)_1 = 0, \quad \text{and} \quad (Tx)_{k+1} = a_k x_k, \quad k = 1, 2, \ldots$$

(a) Find all eigenvalues $\lambda \in \mathbb{C}$ of T satisfying $|\lambda| \ne R$.

(b) Find all eigenvalues $\lambda \in \mathbb{C}$ of T^* satisfying $|\lambda| \ne R$.

(c) Show that the spectral radius $r(T) = \max\{|\lambda| : \lambda \in \sigma(T)\} = R$.

4. Let X be a separable Banach space and let $C \subset X^*$ be a closed convex subset of the dual space. Prove that for any $\ell \in X^*$ there is a point $z \in C$ closest to ℓ, in the sense that

$$\|z - \ell\|_{X^*} \le \|y - \ell\|_{X^*} \quad \text{for all } y \in C.$$

5. Let (x_n) be a sequence in an arbitrary Banach space X and assume $x_n \rightharpoonup x$ as $n \to \infty$. Prove that there exists a sequence (y_n) with $y_n \in \operatorname{span}\{x_1, \ldots, x_n\}$ such that $\|y_n - x\| \to 0$ as $n \to \infty$.

6. If $T\colon X \to Y$ is a map between Banach spaces, let us say T *preserves weak convergence* if for every weakly convergence sequence $x_n \rightharpoonup x$ in X, the sequence $Tx_n \rightharpoonup Tx$ in Y. Prove that for any linear map $T\colon X \to Y$, T is continuous if and only if T preserves weak convergence.